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A Lagrangian direct-interaction approximation for homogeneous isotropic turbulence

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A set of integro-differential equations in the Lagrangian renormalized approximation (Kaneda 1981) is rederived by applying a perturbation method developed by Kraichnan (1959), which is based upon an extraction of direct interactions among Fourier modes of a velocity field and was applied to the Eulerian velocity correlation and response functions, to the Lagrangian ones for homogeneous isotropic turbulence. The resultant set of integro-differential equations for these functions has no adjustable free parameters. The shape of the energy spectrum function is determined numerically in the universal range for stationary turbulence, and in the whole wavenumber range in a similarly evolving form for the freely decaying case. The energy spectrum in the universal range takes the same shape in both cases, which also agrees excellently with many measurements of various kinds of real turbulence as well as numerical results obtained by Gotoh *et al.* (1988) for a decaying case as an initial value problem. The skewness factor of the longitudinal velocity derivative is calculated to be −0.66 for stationary turbulence. The wavenumber dependence of the eddy viscosity is also determined.

1. Introduction

One of the main objectives of the statistical theory of turbulence is to derive the statistical averages of field quantities, such as the mean velocity distribution, the velocity correlation (or the energy spectrum) function, etc., systematically from first principles, i.e. the Navier–Stokes equation. It has long been known, however, that the equations for velocity moments are never closed by simple averaging procedures because of nonlinearity with respect to the velocity of the governing equations. There is a long history of attacking this closure problem. We refer the readers to standard textbooks (e.g. Leslie 1973; Monin & Yaglom 1975; Orszag 1977; Lesieur 1990; McComb 1990) for details. We make here a few remarks only on some theories closely related to the current work.

A lot of closure theories have been proposed so far which are based upon the quasi-normality of the one-point statistics of a turbulent velocity field. The zero-fourth-order cumulant theory is one of the simplest approximations which unfortunately leads to a negative energy spectrum (Monin & Yaglom 1975). The EDQNM (eddy-damped quasi-normal Markovian) approximation (Orszag 1977) guarantees the positivity of the energy spectrum in the inertial range. This theory, however, requires an adjustable free parameter to get a reasonable value of the Kolmogorov constant. An analytical theory of turbulence which does not include any free parameters was developed by Kraichnan (1959, 1965). He proposed the DIA (direct-interaction approximation) and applied it to the Eulerian velocity field, which we call here the Eulerian DIA. However, it does not give the proper $-\frac{5}{3}$ power law of the energy spectrum in the inertial range but a $-\frac{3}{2}$ power law. This failure may be attributed to the non-invariance under Galilean transformation of the Eulerian velocity covariance (Kraichnan 1965; Orszag 1977). Later, he introduced the Lagrangian velocity field and rewrote the Eulerian DIA equations in terms of it and succeeded in obtaining the $-\frac{5}{3}$ power energy spectrum (Kraichnan 1965), which we call Kraichnan's Lagrangian DIA. The analysis is, however, too complicated for the present authors to understand.

The Eulerian DIA equations can also be derived by various kinds of renormalized expansions with respect to the Reynolds number, which include a diagrammatic technique (Wyld 1961; Martin, Siggia & Rose 1973), a primitive Reynolds number expansion followed by a formal change of variables (Leslie 1973) and the reversed expansion method (Kraichan 1977). It should be stressed, however, that the DIA and the above renormalized expansions are based upon completely different ideas of approximation though the resulting equations happen to be same (which is not trivial *a priori*). This coincidence suggests the wide applicability of the results of these approximations. The reversed expansion method was applied to the Lagrangian velocity field by Kaneda (1981) after introducing a mapping function (Lagrangian position function) which relates the Lagrangian and the Eulerian fields and he derived integro-differential equations which he called the LRA (Lagrangian renormalized approximation) equations. They are much simpler than Kraichnan's Lagrangian DIA equations and lead to the $-\frac{5}{3}$ power inertial-range energy spectrum with Kolmogorov constant 1.722. This theory also has no free parameters. The properties of this set of equations have been extensively investigated by himself and his coworkers (Kaneda 1986, 1993; Gotoh, Kaneda & Bekki 1988).

The wrong predictions of the $-\frac{5}{3}$ power spectrum by the Eulerian DIA equation were also corrected by introducing a propagator function relating a two-time Eulerian velocity correlation function with a single-time one (McComb, Shanmugasundaram & Hutchinson 1989; McComb, Filipiak & Shanmugasundaram 1992). Qian (1983) obtained the $-\frac{5}{3}$ power law of the energy spectrum with the Kolmogorov constant 1.2 by a mean-field approach.

The purpose of this paper is twofold. First, we apply Kraichnan's idea of DIA to the Lagrangian field quantities, using the Lagrangian position function, to obtain a set of integro-differential equations for the Lagrangian velocity correlation and the response functions, which happens to be exactly same as the one derived by Kaneda (1981). Then, we seek the shape of the energy spectrum function in two cases, i.e. stationary and freely decaying. In the former the energy spectrum is obtained in the universal range, while in the latter it is determined in the whole wavenumber range in a similarly decaying form. We solve the governing equations by an iteration method instead of the initial value approach used before by Gotoh *et al.* (1988).

This paper is organized as follows. In the next section we introduce several basic quantities which are necessary in the subsequent analysis. In §3, we explain the idea, the assumptions and the procedure for the present Lagrangian DIA and derive a set of integro-differential equations for homogeneous isotropic turbulence. We then solve these equations for a stationary case in §4. The skewness of the velocity derivative and the wavenumber dependence of the eddy viscosity are also calculated. A freely decaying case is treated in §5. The shape of the energy spectrum and the time development are determined in a similarly evolving form. Section 6 is devoted to a discussion. Details of some calculations are given in Appendices.

2. Preparations

2.1. *Basic quantities*

We deal with the motion of an incompressible viscous fluid which is described by the Navier–Stokes equation

$$\frac{\partial}{\partial t} u_i(\boldsymbol{x},t) + u_j(\boldsymbol{x},t)\frac{\partial}{\partial x_j} u_i(\boldsymbol{x},t) = -\frac{1}{\rho}\frac{\partial}{\partial x_i} p(\boldsymbol{x},t) + \nu \frac{\partial^2}{\partial x_j \partial x_j} u_i(\boldsymbol{x},t) \quad (i=1,2,3) \quad (2.1)$$

and continuity equation

$$\frac{\partial}{\partial x_i} u_i(\boldsymbol{x},t) = 0 \ , \quad (2.2)$$

where $u_i(\boldsymbol{x},t)$ is the Eulerian velocity at position $\boldsymbol{x}$ at time t, ρ is the constant density, $p(\boldsymbol{x},t)$ is the pressure and ν is the kinematic viscosity of the fluid. Repeated subscripts are summed over 1–3.

The Lagrangian position function

$$\psi(\boldsymbol{x},t|\boldsymbol{x}',t') = \delta^3\big(\boldsymbol{x} - \boldsymbol{y}(t|\boldsymbol{x}',t')\big) \ , \quad (2.3)$$

which was introduced before by Kaneda (1981) plays an important role in the present analysis, where $\boldsymbol{y}(t|\boldsymbol{x}',t')$ stands for the Lagrangian coordinate (i.e. the position of a fluid element at time t which passed position $\boldsymbol{x}'$ at time t' $(< t)$), and δ^3 is Dirac's delta function. Obviously, the position function obeys

$$\frac{\partial}{\partial t}\psi(\boldsymbol{x},t|\boldsymbol{x}',t') = -u_j(\boldsymbol{x},t)\frac{\partial}{\partial x_j}\psi(\boldsymbol{x},t|\boldsymbol{x}',t') \quad (2.4)$$

with initial condition

$$\psi(\boldsymbol{x},t'|\boldsymbol{x}',t') = \delta^3(\boldsymbol{x}-\boldsymbol{x}') \ . \quad (2.5)$$

The Lagrangian velocity $v_i(t|\boldsymbol{x}',t') = u_i(\boldsymbol{y}(t|\boldsymbol{x}',t'),t)$ and the Eulerian velocity are related to each other as

$$v_i(t|\boldsymbol{x}',t') = \int \mathrm{d}^3\boldsymbol{x}\, u_i(\boldsymbol{x},t)\,\psi(\boldsymbol{x},t|\boldsymbol{x}',t') \ , \quad (2.6)$$

$$u_i(\boldsymbol{x},t) = \int \mathrm{d}^3\boldsymbol{x}'\, v_i(t|\boldsymbol{x}',t')\psi(\boldsymbol{x},t|\boldsymbol{x}',t') \ . \quad (2.7)$$

As mentioned in the introduction, we will construct a system of equations for the Lagrangian velocity correlation function which is defined by

$$V_{ij}(\boldsymbol{r},t,t') = \overline{v_i(t|\boldsymbol{x}+\boldsymbol{r},t')\,u_j(\boldsymbol{x},t')} \ . \quad (2.8)$$

Here and below, an overbar denotes an ensemble average. We have assumed that the velocity field is statistically homogeneous, so that V_{ij} is independent of position vector $\boldsymbol{x}$.

2.2. *Fourier decomposition*

For simplicity of description, we consider the motion of a fluid confined in a periodic cube of side L. Then we can expand u_i, v_i, ψ and V_{ij} in Fourier series as

$$u_i(\boldsymbol{x},t) = \left(\frac{2\pi}{L}\right)^3 \sum_{\boldsymbol{k}} \widetilde{u}_i(\boldsymbol{k},t)\, \exp\big[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}\big] \ , \quad (2.9)$$

$$v_i(t|\boldsymbol{x}',t') = \left(\frac{2\pi}{L}\right)^3 \sum_{\boldsymbol{k}} \widetilde{v}_i(t|\boldsymbol{k},t')\,\exp\Big[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'\Big]\,, \tag{2.10}$$

$$\psi(\boldsymbol{x},t|\boldsymbol{x}',t') = \left(\frac{2\pi}{L}\right)^6 \sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'} \widetilde{\psi}(\boldsymbol{k},t|\boldsymbol{k}',t')\,\exp\Big[\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{x}+\boldsymbol{k}'\cdot\boldsymbol{x}')\Big] \tag{2.11}$$

and

$$V_{ij}(\boldsymbol{r},t,t') = \left(\frac{2\pi}{L}\right)^3 \sum_{\boldsymbol{k}} \widetilde{V}_{ij}(\boldsymbol{k},t,t')\,\exp\Big[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}\Big], \tag{2.12}$$

respectively, where

$$\boldsymbol{k} = \frac{2\pi}{L}\,(n_1,n_2,n_3) \qquad (\,n_1,n_2,n_3 = 0,\pm1,\pm2,\cdots\,) \tag{2.13}$$

is the wavenumber vector. The summations are taken over triplets of integers n_1, n_2 and n_3. The Fourier inverse transformations are similarly written as

$$\widetilde{u}_i(\boldsymbol{k},t) = \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3\boldsymbol{x}\,u_i(\boldsymbol{x},t)\,\exp\Big[-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}\Big]\,, \tag{2.14}$$

$$\widetilde{v}_i(t|\boldsymbol{k},t') = \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3\boldsymbol{x}\,v_i(t|\boldsymbol{x}',t')\,\exp\Big[-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}'\Big]\,, \tag{2.15}$$

$$\widetilde{\psi}(\boldsymbol{k},t|\boldsymbol{k}',t') = \left(\frac{1}{2\pi}\right)^6 \int \mathrm{d}^3\boldsymbol{x} \int \mathrm{d}^3\boldsymbol{x}'\,\psi(\boldsymbol{x},t|\boldsymbol{x}',t')\,\exp\Big[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}+\boldsymbol{k}'\cdot\boldsymbol{x}')\Big] \tag{2.16}$$

and

$$\begin{aligned}\widetilde{V}_{ij}(\boldsymbol{k},t,t') &= \left(\frac{1}{2\pi}\right)^3 \int \mathrm{d}^3\boldsymbol{r}\,V_{ij}(\boldsymbol{r},t,t')\,\exp\Big[-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}\Big] \\ &= \left(\frac{2\pi}{L}\right)^3 \overline{\widetilde{v}_i(t|\boldsymbol{k},t')\,\widetilde{u}_j(-\boldsymbol{k},t')}\,,\end{aligned} \tag{2.17}$$

respectively. In these equations, integrations are carried out over the periodic cube. Relations between the Eulerian velocity and Lagrangian velocity (2.6) and (2.7) are written, in Fourier space, as

$$\widetilde{v}_i(t|\boldsymbol{k},t') = \frac{(2\pi)^6}{L^3} \sum_{\boldsymbol{k}'} \widetilde{u}_i(\boldsymbol{k}',t)\,\widetilde{\psi}(-\boldsymbol{k}',t|\boldsymbol{k},t') \tag{2.18}$$

and

$$\widetilde{u}_i(\boldsymbol{k},t) = \frac{(2\pi)^6}{L^3} \sum_{\boldsymbol{k}'} \widetilde{v}_i(t|\boldsymbol{k}',t')\,\widetilde{\psi}(\boldsymbol{k},t|-\boldsymbol{k}',t')\,, \tag{2.19}$$

respectively.

The governing equations for $\widetilde{u}_i$ and $\widetilde{\psi}$ are derived from (2.1), (2.2), (2.4) and (2.5) as

$$\left[\frac{\partial}{\partial t} + \nu\,k^2\right]\widetilde{u}_i(\boldsymbol{k},t) = -\frac{\mathrm{i}}{2}\left(\frac{2\pi}{L}\right)^3 \widetilde{P}_{ijm}(\boldsymbol{k}) \sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}}_{(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})} \widetilde{u}_j(-\boldsymbol{p},t)\,\widetilde{u}_m(-\boldsymbol{q},t)\,, \tag{2.20}$$

$$k_i\,\widetilde{u}_i(\boldsymbol{k},t) = 0\,, \tag{2.21}$$

and

$$\frac{\partial}{\partial t}\,\widetilde{\psi}(\boldsymbol{k},t|\boldsymbol{k}',t') = -\mathrm{i}\,k_j\left(\frac{2\pi}{L}\right)^3 \sum_{\boldsymbol{p}}\sum_{\substack{\boldsymbol{q}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}} \widetilde{u}_j(-\boldsymbol{p},t)\,\widetilde{\psi}(-\boldsymbol{q},t|\boldsymbol{k}',t') \tag{2.22}$$

with initial condition

$$\widetilde{\psi}(\boldsymbol{k},t'|\boldsymbol{k}',t') = \frac{L^3}{(2\pi)^6}\,\delta^3_{\boldsymbol{k}+\boldsymbol{k}'}\ . \tag{2.23}$$

Here,

$$\widetilde{P}_{ijm}(\boldsymbol{k}) = k_m\,\widetilde{P}_{ij}(\boldsymbol{k}) + k_j\,\widetilde{P}_{im}(\boldsymbol{k}),\quad \widetilde{P}_{ij}(\boldsymbol{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}\ , \tag{2.24}$$

where $\delta^3_{\boldsymbol{k}}$ and δ_{ij} are Kronecker's deltas ($\delta^3_{\boldsymbol{k}} = 0\,(\boldsymbol{k}\neq\boldsymbol{0})$, $\delta^3_{\boldsymbol{k}} = 1\,(\boldsymbol{k}=\boldsymbol{0})$, $\delta_{ij} = 0\,(i\neq j)$, $\delta_{ij} = 1\,(i=j)$). The time-derivative of (2.18) then yields

$$\begin{aligned}\frac{\partial}{\partial t}\,\widetilde{v}_i(t|\boldsymbol{k}',t') = &-\frac{(2\pi)^6}{L^3}\,\nu\sum_{\boldsymbol{p}} p^2\,\widetilde{u}_i(\boldsymbol{p},t)\,\widetilde{\psi}(-\boldsymbol{p},t|\boldsymbol{k}',t')\\ &-\mathrm{i}\,\frac{(2\pi)^9}{L^6}\sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}}\sum_{\substack{\boldsymbol{r}\\(\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0})}}\frac{r_i r_m r_n}{r^2}\,\widetilde{u}_m(\boldsymbol{p},t)\,\widetilde{u}_n(\boldsymbol{q},t)\,\widetilde{\psi}(\boldsymbol{r},t|\boldsymbol{k}',t')\ ,\end{aligned} \tag{2.25}$$

where use has been made of (2.20) and (2.22). By using (2.17), (2.20) and (2.25), we can derive the governing equations of the two-point Lagrangian velocity correlation function for a single time as

$$\begin{aligned}\left[\frac{\partial}{\partial t} + 2\nu k^2\right]\widetilde{V}_{ij}(\boldsymbol{k},t,t) = &-\frac{\mathrm{i}}{2}\left(\frac{2\pi}{L}\right)^6\widetilde{P}_{imn}(\boldsymbol{k})\sum_{\boldsymbol{p}}\sum_{\substack{\boldsymbol{q}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\overline{\widetilde{u}_m(-\boldsymbol{p},t)\,\widetilde{u}_n(-\boldsymbol{q},t)\,\widetilde{u}_j(-\boldsymbol{k},t)}\\ &+(i\leftrightarrow j\ ,\ \boldsymbol{k}\rightarrow-\boldsymbol{k})\end{aligned} \tag{2.26}$$

and for two times as

$$\begin{aligned}\frac{\partial}{\partial t}\,\widetilde{V}_{ij}(\boldsymbol{k},t,t') = &-\frac{(2\pi)^9}{L^6}\,\nu\sum_{\boldsymbol{p}} p^2\,\overline{\widetilde{u}_i(\boldsymbol{p},t)\,\widetilde{\psi}(-\boldsymbol{p},t|\boldsymbol{k},t')\,\widetilde{u}_j(-\boldsymbol{k},t')}\\ &-\mathrm{i}\,\frac{(2\pi)^{12}}{L^9}\sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}}\sum_{\substack{\boldsymbol{r}\\(\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0})}}\frac{r_i r_m r_n}{r^2}\,\overline{\widetilde{u}_m(\boldsymbol{p},t)\,\widetilde{u}_n(\boldsymbol{q},t)\,\widetilde{\psi}(\boldsymbol{r},t|\boldsymbol{k},t')\,\widetilde{u}_j(-\boldsymbol{k},t')}\ .\end{aligned} \tag{2.27}$$

Notice that the above two equations contain three- and four-point correlation functions. Similarly the governing equation of the three-point velocity correlation function would contain four- and five-point correlation functions. Therefore the system of equations cannot be closed, unless multi-point (e.g. four-point) correlation functions are expressed in terms of fewer-point (e.g. two-point) ones at some stage.

For later convenience, we define here the energy spectrum

$$E(k,t) = \tfrac{1}{2}\,k^2\oint \mathrm{d}\Omega\ \widetilde{V}_{ii}(\boldsymbol{k},t,t)\ , \tag{2.28}$$

where $\oint \mathrm{d}\Omega$ denotes the solid angle integration in the Fourier space, and the incompressible part of the Lagrangian velocity correlation

$$\widetilde{Q}_{ij}(\boldsymbol{k},t,t') = \widetilde{P}_{im}(\boldsymbol{k})\,\widetilde{V}_{mj}(\boldsymbol{k},t,t')\ . \tag{2.29}$$

2.3. *Response function*

The response functions of the velocity and the position functions play a key role in the present formulation (Kraichnan 1959; Kaneda 1981). The Eulerian velocity response function

$$\widetilde{G}^{(E)}_{ij}(\boldsymbol{k},t|\boldsymbol{k}',t') = \frac{\delta\widetilde{u}_i(\boldsymbol{k},t)}{\delta\widetilde{u}_j(\boldsymbol{k}',t')} \tag{2.30}$$

expresses the influence on $\widetilde{u}_i(\boldsymbol{k},t)$ at time t due to an infinitesimal disturbance for $\widetilde{u}_j(\boldsymbol{k}',t')$ $(t' \leqslant t)$, where δ denotes a functional derivative. By taking a functional derivative of (2.20), we obtain the governing equation for this function as

$$\left[\frac{\partial}{\partial t} + \nu k^2\right]\widetilde{G}^{(E)}_{ij}(\boldsymbol{k},t|\boldsymbol{k}',t') = -\mathrm{i}\left(\frac{2\pi}{L}\right)^3 \widetilde{P}_{imn}(\boldsymbol{k}) \sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}}_{(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})} \widetilde{u}_m(-\boldsymbol{p},t)\,\widetilde{G}^{(E)}_{nj}(-\boldsymbol{q},t|\boldsymbol{k}',t')\,. \tag{2.31}$$

The initial condition is given by

$$\widetilde{G}^{(E)}_{ij}(\boldsymbol{k},t'|\boldsymbol{k}',t') = \frac{L^3}{(2\pi)^6}\,\delta_{ij}\,\delta^3_{\boldsymbol{k}+\boldsymbol{k}'}\,. \tag{2.32}$$

Similarly, the governing equations of the Lagrangian velocity response function

$$\widetilde{G}^{(L)}_{ij}(t|\boldsymbol{k},\boldsymbol{k}',t') = \frac{\delta\widetilde{v}_i(t|\boldsymbol{k},t')}{\delta\widetilde{u}_j(\boldsymbol{k}',t')} \tag{2.33}$$

and the position response function

$$\widetilde{\Psi}_i(\boldsymbol{k},t|\boldsymbol{k}',\boldsymbol{k}'',t') = \frac{\delta\widetilde{\psi}(\boldsymbol{k},t|\boldsymbol{k}',t')}{\delta\widetilde{u}_i(\boldsymbol{k}'',t')} \tag{2.34}$$

are obtained from (2.25) and (2.22), respectively, as

$$\begin{aligned}
&\frac{\partial}{\partial t}\,\widetilde{G}^{(L)}_{ij}(t|\boldsymbol{k},\boldsymbol{k}',t') \\
&= -\nu\,\frac{(2\pi)^6}{L^3}\sum_{\boldsymbol{k}''} k''^2\left[\widetilde{G}^{(E)}_{ij}(\boldsymbol{k}'',t|\boldsymbol{k}',t')\,\widetilde{\psi}(-\boldsymbol{k}'',t|\boldsymbol{k},t') + \widetilde{u}_i(\boldsymbol{k}'',t)\,\widetilde{\Psi}_j(-\boldsymbol{k}'',t|\boldsymbol{k},\boldsymbol{k}',t')\right] \\
&\quad -\mathrm{i}\,\frac{(2\pi)^9}{L^6}\sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{r}}_{(\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0})}\frac{r_i r_m r_n}{r^2} \\
&\quad \times\left[2\widetilde{u}_m(\boldsymbol{p},t)\,\widetilde{G}^{(E)}_{nj}(\boldsymbol{q},t|\boldsymbol{k}',t')\,\widetilde{\psi}(\boldsymbol{r},t|\boldsymbol{k},t') + \widetilde{u}_m(\boldsymbol{p},t)\,\widetilde{u}_n(\boldsymbol{q},t)\,\widetilde{\Psi}_j(\boldsymbol{r},t|\boldsymbol{k},\boldsymbol{k}',t')\right]
\end{aligned} \tag{2.35}$$

with initial condition

$$\widetilde{G}^{(L)}_{ij}(t'|\boldsymbol{k},\boldsymbol{k}',t') = \frac{L^3}{(2\pi)^6}\,\delta_{ij}\,\delta^3_{\boldsymbol{k}+\boldsymbol{k}'}\,, \tag{2.36}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t}\,\widetilde{\Psi}_i(\boldsymbol{k},t|\boldsymbol{k}',\boldsymbol{k}'',t') = -\mathrm{i}\,k_a\left(\frac{2\pi}{L}\right)^3 \\
&\quad\times\sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}}_{(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}\left[\widetilde{u}_a(-\boldsymbol{p},t)\,\widetilde{\Psi}_i(-\boldsymbol{q},t|\boldsymbol{k}',\boldsymbol{k}'',t') + \widetilde{G}^{(E)}_{ai}(-\boldsymbol{p},t|\boldsymbol{k}'',t')\,\widetilde{\psi}(-\boldsymbol{q},t|\boldsymbol{k}',t')\right]
\end{aligned} \tag{2.37}$$

with initial condition

$$\widetilde{\Psi}_i(\boldsymbol{k}, t'|\boldsymbol{k}', \boldsymbol{k}'', t') = 0 \ . \tag{2.38}$$

A functional derivative of (2.19) gives a relation among the Eulerian velocity response, the Lagrangian velocity response and the position response functions as

$$\begin{aligned}\widetilde{G}^{(E)}_{ij}(\boldsymbol{k}, t|\boldsymbol{k}', t') = {} & \frac{(2\pi)^6}{L^3} \sum_{\boldsymbol{k}''} \widetilde{G}^{(L)}_{ij}(t|\boldsymbol{k}'', \boldsymbol{k}', t')\, \widetilde{\psi}(\boldsymbol{k}, t| - \boldsymbol{k}'', t') \\ & + \frac{(2\pi)^{12}}{L^6} \sum_{\boldsymbol{k}''} \sum_{\boldsymbol{k}'''} \widetilde{u}_i(\boldsymbol{k}''', t)\, \widetilde{\psi}(-\boldsymbol{k}''', t|\boldsymbol{k}'', t')\, \widetilde{\Psi}_j(\boldsymbol{k}, t| - \boldsymbol{k}'', \boldsymbol{k}', t') \ . \end{aligned} \tag{2.39}$$

For later use, we define here the incompressible part of the Lagrangian velocity response function by

$$\widetilde{G}_{ij}(\boldsymbol{k}, t, t') = \frac{(2\pi)^6}{L^3} \overline{\widetilde{G}^{(L)}_{im}(t|\boldsymbol{k}, -\boldsymbol{k}, t')}\, \widetilde{P}_{mj}(\boldsymbol{k}) \ . \tag{2.40}$$

2.4. *Homogeneous isotropic turbulence*

If turbulence is isotropic as well as homogeneous, second-order tensors $\widetilde{Q}_{ij}$ and $\widetilde{G}_{ij}$ are represented by a single scalar as

$$\widetilde{Q}_{ij}(\boldsymbol{k}, t, t') = \tfrac{1}{2}\, \widetilde{P}_{ij}(\boldsymbol{k})\, Q(k, t, t') \ , \tag{2.41}$$

and

$$\widetilde{G}_{ij}(\boldsymbol{k}, t, t') = \widetilde{P}_{ij}(\boldsymbol{k})\, G(k, t, t') \ . \tag{2.42}$$

Then, the energy spectrum is represented, from (2.28), (2.29) and (2.41), as

$$E(k, t) = 2\pi k^2\, Q(k, t, t) \ . \tag{2.43}$$

The energy spectrum at large wavenumbers for (locally) homogeneous isotropic and stationary turbulence at large Reynolds numbers was considered by Kolmogorov (1941). By employing a dimensional analysis in terms of wavenumber k, kinematic viscosity ν and the mean of energy dissipation rate ϵ, he derived a similarity form of the energy spectrum function at large wavenumbers as

$$E(k, t) = \nu^{5/4}\, \epsilon^{1/4}\, F(k/k_d) \ , \tag{2.44}$$

$$k_d = \epsilon^{1/4} \nu^{-3/4} \ , \tag{2.45}$$

where F is a non-dimensional function and k_d is the Kolmogorov wavenumber. The wavenumber range in which similarity law (2.44) is observed is called the universal range. This similarity law has been supported by many measurements of different kinds of turbulence (see Chapman 1979; Saddoughi & Veeravelli 1994 and also figure 2). In the inertial subrange ($k \ll k_d$), the spectrum may not depend on ν and takes power form

$$E(k, t) = K\, \epsilon^{2/3}\, k^{-5/3} \ , \tag{2.46}$$

where K is the Kolmogorov constant, which is evaluated experimentally to be 1.62 ± 0.17 (Sreenivasan 1995). One of the main objectives of the current closure theory of turbulence is to determine the universal function F with resort to the Navier–Stokes equation.

3. Lagrangian direct-interaction approximation

3.1. *Direct-interaction decomposition*

Following an idea developed by Kraichnan (1959), we introduce a direct-interaction decomposition. Recall that the right-hand side of Navier–Stokes equation (2.20) is composed of a sum of an infinite number of quadratic nonlinear terms, each of which represents direct interactions among three Fourier components with wavenumbers $\boldsymbol{k}$, $\boldsymbol{p}$ and $\boldsymbol{q}$ which construct a triangle ($\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}$).

Here, we choose arbitrarily a triangular triplet of wavenumbers, say $\boldsymbol{k}_0$, $\boldsymbol{p}_0$ and $\boldsymbol{q}_0$ ($\boldsymbol{k}_0+\boldsymbol{p}_0+\boldsymbol{q}_0=\boldsymbol{0}$). We imagine then a fictitious field which does not contain the direct interactions among these special wavenumbers. This fictitious field is called the NDI-field (non-direct-interaction field), and is denoted by $\widetilde{u}_i^{(0)}(\boldsymbol{k},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$. Furthermore, we define the DI-field (direct-interaction field) $\widetilde{u}_i^{(1)}(\boldsymbol{k},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$ by the difference between the true field and the NDI-field, namely

$$\widetilde{u}_i(\boldsymbol{k},t)=\widetilde{u}_i^{(0)}(\boldsymbol{k},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)+\widetilde{u}_i^{(1)}(\boldsymbol{k},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0). \tag{3.1}$$

Hence, the governing equation for $\widetilde{u}_i^{(0)}$ is written as

$$\left[\frac{\partial}{\partial t}+\nu k^2\right]\widetilde{u}_i^{(0)}(\boldsymbol{k},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$$
$$=-\frac{\mathrm{i}}{2}\left(\frac{2\pi}{L}\right)^3\widetilde{P}_{ijm}(\boldsymbol{k})\sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}}{}'_{(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}\widetilde{u}_j^{(0)}(-\boldsymbol{p},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\widetilde{u}_m^{(0)}(-\boldsymbol{q},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\ , \tag{3.2}$$

where $\Sigma\Sigma'$ stands for summation without the interactions among chosen three wavenumbers $\boldsymbol{k}_0$, $\boldsymbol{p}_0$ and $\boldsymbol{q}_0$. Subtracting the above equation from the Navier–Stokes equation, we obtain the time-evolution equation of the DI-field as

$$\left[\frac{\partial}{\partial t}+\nu k^2\right]\widetilde{u}_i^{(1)}(\boldsymbol{k},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$$
$$=-\mathrm{i}\left(\frac{2\pi}{L}\right)^3\widetilde{P}_{ijm}(\boldsymbol{k})\sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}}{}'_{(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}\widetilde{u}_j(-\boldsymbol{p},t)\widetilde{u}_m^{(1)}(-\boldsymbol{q},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$$
$$-\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0}\,\mathrm{i}\left(\frac{2\pi}{L}\right)^3\widetilde{P}_{ijm}(\boldsymbol{k}_0)\ \widetilde{u}_j^{(0)}(-\boldsymbol{p}_0,t|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{u}_m^{(0)}(-\boldsymbol{q}_0,t|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$$
$$+\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0}\,\mathrm{i}\left(\frac{2\pi}{L}\right)^3\widetilde{P}_{ijm}(\boldsymbol{k}_0)\,\widetilde{u}_j^{(0)}(\boldsymbol{p}_0,t|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{u}_m^{(0)}(\boldsymbol{q}_0,t|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$$
$$+(\boldsymbol{p}_0\rightarrow\boldsymbol{q}_0\rightarrow\boldsymbol{k}_0\rightarrow\boldsymbol{p}_0)\ . \tag{3.3}$$

Here we have neglected the higher-order terms of $\widetilde{u}^{(1)}$ (Assumption 1 written in the next section). The direct-interaction decompositions for the Eulerian velocity response function (2.31), position function (2.22) and position response function (2.37) are made similarly. The details of the calculation are described in Appendix A.

By comparing (3.3) with (A 4) in Appendix A, we find that $\widetilde{G}_{ij}^{(E0)}$ serves as Green's function of $\widetilde{u}_i^{(1)}$. A solution to (3.3), therefore, is expressed in terms of $\widetilde{G}_{ij}^{(E0)}$ and $\widetilde{u}_i^{(0)}$ as

$$\widetilde{u}_i^{(1)}(\boldsymbol{k},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0) = \mathrm{i}\,\frac{(2\pi)^9}{L^6}\,\widetilde{P}_{abc}(\boldsymbol{k}) \int_{t_0}^{t} \mathrm{d}t'\, \widetilde{G}_{ia}^{(E0)}(\boldsymbol{k},t|-\boldsymbol{k},t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$$
$$\times\big[-\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0}\,\widetilde{u}_b^{(0)}(-\boldsymbol{p}_0,t'|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{u}_c^{(0)}(-\boldsymbol{q}_0,t'|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$$
$$-\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0}\,\widetilde{u}_b^{(0)}(\boldsymbol{p}_0,t'|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{u}_c^{(0)}(\boldsymbol{q}_0,t'|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$$
$$+(\boldsymbol{p}_0\to\boldsymbol{q}_0\to\boldsymbol{k}_0\to\boldsymbol{p}_0)\big], \quad (3.4)$$

where $\widetilde{u}_i^{(1)}$ has been assumed to vanish at time t_0. The DI-fields of other quantities are similarly represented in terms of the NDI-fields (see Appendix B).

3.2. *Methods of approximation*

For the purpose of easy reference, we summarize here assumptions and procedures to construct a system of equations for statistical quantities (i.e. the Lagrangian velocity correlation and the response functions). We choose a coordinate system with zero mean velocity and make the following three assumptions (Kraichnan 1959):

Assumption 1 $\widetilde{X}^{(1)}$ is much smaller than $\widetilde{X}^{(0)}$ in magnitude, hence we can neglect their higher-order terms. Here, $\widetilde{X}$ stands for any physical quantity (for example $\widetilde{u}_i$).

Assumption 2 $\widetilde{u}_i^{(0)}(\boldsymbol{k}_0,t||\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$, $\widetilde{u}_j^{(0)}(\boldsymbol{p}_0,t'||\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$ and $\widetilde{u}_k^{(0)}(\boldsymbol{q}_0,t''||\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$ are statistically independent of each other.

Assumption 3 $\{\widetilde{u}\}$, $\{\widetilde{G}\}$, $\{\widetilde{\psi}\}$ and $\{\widetilde{\Psi}\}$ are statistically independent of each other.

We construct a set of integro-differential equations for the velocity correlation and the response functions by employing the following four procedures :

Procedure 1 Substitute the direct-interaction decompositions, (3.1) and (A 1)–(A 3), into the right-hand side of the governing equations of statistical quantities. Thanks to Assumption 1, we can neglect higher-order terms of the DI-field.

Procedure 2 Eliminate physical quantities of the DI-field by making use of (3.4), (B 1), (B 2) and (B 3).

Procedure 3 Eliminate $\overline{\widetilde{G}_{ij}^{(E0)}}$, $\overline{\widetilde{\psi}^{(0)}}$ and $\overline{\widetilde{\Psi}_i^{(0)}}$ respectively by making use of

$$\overline{\widetilde{G}_{ij}^{(E0)}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)} = \overline{\widetilde{G}_{ij}^{(L0)}(t|\boldsymbol{k},\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}\,, \quad (3.5)$$

$$\overline{\widetilde{\psi}^{(0)}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)} = \frac{L^3}{(2\pi)^6}\,\delta^3_{\boldsymbol{k}+\boldsymbol{k}'} \quad (3.6)$$

and

$$\overline{\widetilde{\Psi}_i^{(0)}(\boldsymbol{k},t|\boldsymbol{k}',\boldsymbol{k}'',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)} = -\frac{\mathrm{i}\,k_a}{(2\pi)^3}\int_{t'}^{t}\mathrm{d}t''\,\overline{\widetilde{G}_{ai}^{(L0)}(t''|\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{k}'',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}\,. \quad (3.7)$$

Procedure 4 Replace $\overline{\widetilde{u}_i^{(0)}\widetilde{u}_j^{(0)}}$ and $\overline{\widetilde{G}_{im}^{(L0)}}$ by $\widetilde{Q}_{ij}$ and $\widetilde{G}_{ij}$ through

$$\overline{\widetilde{u}_i^{(0)}(\boldsymbol{k},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{u}_j^{(0)}(-\boldsymbol{k},t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)} = \left(\frac{L}{2\pi}\right)^3\widetilde{Q}_{ij}(\boldsymbol{k},t,t') \quad (3.8)$$

and

$$\frac{(2\pi)^6}{L^3}\,\overline{\widetilde{G}_{im}^{(L0)}(t|\boldsymbol{k},-\boldsymbol{k},t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}\,\widetilde{P}_{mj}(\boldsymbol{k}) = \widetilde{G}_{ij}(\boldsymbol{k},t,t')\,, \quad (3.9)$$

which follows from (2.40) under Assumption 1. For derivations of relations (3.5)–(3.8), see Appendix C.

3.3. *Closed system of equations for statistical quantities*

Applying the methods of the Lagrangian DIA described in the preceding section, we can construct a system of equations for statistical quantities. We start with the two-point one-time velocity correlation function. By substituting direct-interaction decomposition (3.1) into the right-hand side of (2.26), we obtain

$$\left[\frac{\partial}{\partial t}+2\nu k^2\right]\widetilde{V}_{ij}(\boldsymbol{k},t,t)=-\frac{\mathrm{i}}{2}\left(\frac{2\pi}{L}\right)^6\widetilde{P}_{imn}(\boldsymbol{k})$$
$$\times\sum_{\boldsymbol{p}}\sum_{\substack{\boldsymbol{q}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\left[\overline{\widetilde{u}_m^{(0)}(-\boldsymbol{p},t\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\,\widetilde{u}_n^{(0)}(-\boldsymbol{q},t\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\,\widetilde{u}_j^{(0)}(-\boldsymbol{k},t\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})}\right.$$
$$+2\,\overline{\widetilde{u}_m^{(0)}(-\boldsymbol{p},t\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\,\widetilde{u}_n^{(1)}(-\boldsymbol{q},t\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\,\widetilde{u}_j^{(0)}(-\boldsymbol{k},t\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})}$$
$$\left.+\overline{\widetilde{u}_m^{(0)}(-\boldsymbol{p},t\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\,\widetilde{u}_n^{(0)}(-\boldsymbol{q},t\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\,\widetilde{u}_j^{(1)}(-\boldsymbol{k},t\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})}\right]+(i\leftrightarrow j\ ,\ \boldsymbol{k}\to-\boldsymbol{k}). \tag{3.10}$$

Here a set of removed wavenumbers has been selected as $\left(\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0\right)=\left(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}\right)$ in the summand of the right-hand side of the above equation, and higher-order terms of the DI-field have been neglected (Assumption 1). Applying Procedures 2–4 to (3.10), we obtain the time-evolution equation of the two-point velocity correlation function as

$$\left[\frac{\partial}{\partial t}+2\nu k^2\right]\widetilde{Q}_{ij}(\boldsymbol{k},t,t)$$
$$=\frac{1}{2}\left(\frac{2\pi}{L}\right)^3\widetilde{P}_{ia}(\boldsymbol{k})\times\left\{\widetilde{P}_{amn}(\boldsymbol{k})\sum_{\boldsymbol{p}}\sum_{\substack{\boldsymbol{q}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\int_{t_0}^{t}\mathrm{d}t'\,\widetilde{Q}_{mb}(-\boldsymbol{p},t,t')\right.$$
$$\times\left[2\Big(q_b\,\widetilde{G}_{nc}(-\boldsymbol{q},t,t')+q_c\,\widetilde{G}_{nb}(-\boldsymbol{q},t,t')\Big)\,\widetilde{Q}_{jc}(-\boldsymbol{k},t,t')\right.$$
$$\left.\left.+\Big(k_b\,\widetilde{G}_{jc}(-\boldsymbol{k},t,t')+k_c\,\widetilde{G}_{jb}(-\boldsymbol{k},t,t')\Big)\,\widetilde{Q}_{nc}(-\boldsymbol{q},t,t')\right]+(a\leftrightarrow j,\ \boldsymbol{k}\to-\boldsymbol{k})\right\}, \tag{3.11}$$

(see Appendix D for the derivation).

In a similar way, we can express the governing equation of the two-point two-time Lagrangian velocity correlation function in terms of only the two-point Lagrangian velocity correlation and the response functions. Employing Procedures 1–4 for (2.27) (see Appendix E), we obtain

$$\left[\frac{\partial}{\partial t}+\nu k^2\right]\widetilde{Q}_{ij}(\boldsymbol{k},t,t')$$
$$=-2\left(\frac{2\pi}{L}\right)^3\widetilde{P}_{id}(\boldsymbol{k})\sum_{\boldsymbol{p}}\sum_{\substack{\boldsymbol{q}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\frac{q_aq_bq_cq_d}{q^2}\int_{t'}^{t}\mathrm{d}t''\,\widetilde{Q}_{ab}(\boldsymbol{p},t,t'')\,\widetilde{Q}_{cj}(\boldsymbol{k},t,t')\,. \tag{3.12}$$

Finally, for the Lagrangian velocity response function, it follows from (2.35) (see Appendix E) that

$$\left[\frac{\partial}{\partial t}+\nu k^2\right]\widetilde{G}_{ij}(\boldsymbol{k},t,t')=-2\left(\frac{2\pi}{L}\right)^3\sum_{\substack{\boldsymbol{p}\quad\boldsymbol{q}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\!\!\sum\frac{k_ik_mk_n}{k^2}$$
$$\times\int_{t'}^{t}\mathrm{d}t''\left[q_b\widetilde{G}_{nc}(-\boldsymbol{q},t,t')+q_c\widetilde{G}_{nb}(-\boldsymbol{q},t,t')\right]\widetilde{G}_{cj}(\boldsymbol{k},t'',t')\,\widetilde{Q}_{mb}(-\boldsymbol{p},t,t'')$$
$$-2\left(\frac{2\pi}{L}\right)^3\sum_{\substack{\boldsymbol{p}\quad\boldsymbol{q}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\!\!\sum\frac{q_aq_bq_cq_i}{q^2}\int_{t'}^{t}\mathrm{d}t''\,\widetilde{Q}_{ab}(\boldsymbol{p},t,t'')\,\widetilde{G}_{cj}(\boldsymbol{k},t,t')\,. \qquad (3.13)$$

Equations (3.11)–(3.13) constitute a closed system of equations for the time-evolution of the Lagrangian velocity correlation and the response functions. It should be remarked that this system of equations agrees exactly with that obtained before by the use of LRA (Kaneda 1981). It is interesting to observe that completely different methods lead to an identical result, which suggests a wide applicability of the equations.

Equations (3.11)–(3.13) were derived for homogeneous turbulence. If turbulence is isotropic as well as homogeneous, they reduce to

$$\left[\frac{\partial}{\partial t}+2\nu k^2\right]Q(k,t,t)=2\pi\iint_{\Delta_k}\mathrm{d}p\mathrm{d}q\,kpq\,\widehat{b}(k,p,q)$$
$$\times\int_{t_0}^{t}\mathrm{d}t'\,Q(q,t,t')\left[G(k,t,t')\,Q(p,t,t')-G(p,t,t')\,Q(k,t,t')\right], \qquad (3.14)$$

$$\left[\frac{\partial}{\partial t}+\nu k^2+\widehat{\eta}(k,t,t')\right]Q(k,t,t')=0\,, \qquad (3.15)$$

$$\left[\frac{\partial}{\partial t}+\nu k^2+\widehat{\eta}(k,t,t')\right]G(k,t,t')=0 \qquad (3.16)$$

and

$$G(k,t,t)=1 \qquad (3.17)$$

respectively (for the derivation, see Kaneda 1981), where $\iint_{\Delta_k}\mathrm{d}p\mathrm{d}q$ denotes an integration under the condition that k, p and q constitute a triangle. We have taken the limit that the size L of a periodic box is infinity. The content in the rest of this section and §4.1 is essentially the same as that in Kaneda (1981, 1986). However, we recapitulate it because it is necessary for further analysis in the subsequent sections. Equations (4.5) and (4.6) are basic in §4.2, and (3.16), (3.22) and (4.13) are essential for the analysis in §5. Functions $\widehat{b}(k,p,q)$ and $\widehat{\eta}(k,t,t')$ in (3.14)–(3.16) are respectively defined by

$$\widehat{b}(k,p,q)=\frac{1}{8k^4p^2q^2}\left[k^2-(p-q)^2\right]\left[(p+q)^2-k^2\right]\left[(2p^2-q^2)\,k^2-(p^2-q^2)\,q^2\right] \qquad (3.18)$$

and

$$\widehat{\eta}(k,t,t') = \tfrac{4}{3}\,\pi\,k^5 \int_0^\infty \mathrm{d}p\, p^{10/3}\, J(p^{2/3}) \int_{t'}^{t} \mathrm{d}t''\, Q(kp,t,t'') \tag{3.19}$$

with

$$J(p) = \frac{3}{32p^5}\left[\frac{(1-p^3)^4}{2p^{3/2}}\,\log\frac{(1+p^{3/2})}{|1-p^{3/2}|} - \frac{1+p^3}{3}\,(3p^6-14p^3+3)\right]. \tag{3.20}$$

It follows from (3.15)–(3.17) that

$$Q(k,t,t') = Q(k,t',t')\,G(k,t,t')\,. \tag{3.21}$$

Once the single-time velocity correlation function $Q(k,t,t)$ and the response function $G(k,t,t')$ are determined, the two-time velocity correlation function $Q(k,t,t')$ follows from (3.21). In homogeneous isotropic turbulence, therefore, it is sufficient to deal with the system of equations for $Q(k,t,t)$ and $G(k,t,t')$. Note that using (3.14) and (3.21) we can write the equation for $Q(k,t,t)$ as

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] Q(k,t,t) = 2\pi \iint_{\Delta_k} \mathrm{d}p\mathrm{d}q\, kpq\,\widehat{b}(k,p,q)$$
$$\times \int_{t_0}^{t} \mathrm{d}t'\, G(k,t,t')\,G(p,t,t')\,G(q,t,t')\,Q(q,t',t')\left[Q(p,t',t') - Q(k,t',t')\right]. \tag{3.22}$$

4. Stationary turbulence

In this section, we consider how the resultant equations (3.16) and (3.22) in the previous section behave in homogeneous isotropic stationary turbulence. Under the assumption of stationarity, it is shown that Q and G depend only on the difference between t and t', so that we put

$$Q(k,t,t') = \check{Q}(k,t-t'), \tag{4.1}$$
$$G(k,t,t') = \check{G}(k,t-t')\,. \tag{4.2}$$

Then, the single-time velocity correlation function is written as

$$Q(k,t,t) = \check{Q}(k,0)\,, \tag{4.3}$$

and (3.21) as

$$\check{Q}(k,t) = \check{Q}(k,0)\,\check{G}(k,t)\,. \tag{4.4}$$

Introduction of the above relations into (3.16) yields

$$\frac{\partial}{\partial t}\,\log\check{Q}(k,t) = -\nu k^2 - \tfrac{4}{3}\,\pi\,k^5 \int_0^\infty \mathrm{d}p\, p^{10/3}\, J(p^{2/3}) \int_0^t \mathrm{d}t'\,\check{Q}(kp,t')\,. \tag{4.5}$$

On the other hand, integration from k_0 to infinity with respect to k of (3.22) multiplied by $2\pi k^2$ leads to

$$4\pi\nu \int_0^{k_0} \mathrm{d}k\, k^4\,\check{Q}(k,0) = \epsilon - 4\pi^2 \int_{k_0}^\infty \mathrm{d}k \iint_{\Delta_k} \mathrm{d}p\mathrm{d}q\, k^3 pq\,\widehat{b}(k,p,q)$$
$$\times \int_0^\infty \mathrm{d}t'\,\check{Q}(k,t')\,\check{Q}(p,t')\,\check{Q}(q,t')\left[\check{Q}(k,0)^{-1} - \check{Q}(p,0)^{-1}\right], \tag{4.6}$$

where use has been made of

$$4\pi\nu \int_0^\infty \mathrm{d}k\, k^4\, \check{Q}(k,0) = \epsilon\ , \tag{4.7}$$

and the initial time has been put at $t_0 = -\infty$. Equations (4.5) and (4.6) constitute the basic equations in stationary turbulence.

4.1. *Kolmogorov constant*

To examine the behaviour of these equations in the inertial subrange, we take the limit of $\nu \to 0$. Then, (4.5) and (4.6) are reduced respectively to

$$\frac{\partial}{\partial t} \log \check{Q}(k,t) = -\tfrac{4}{3}\,\pi\, k^5 \int_0^\infty \mathrm{d}p\, p^{10/3}\, J(p^{2/3}) \int_0^t \mathrm{d}t'\, \check{Q}(kp,t') \tag{4.8}$$

and

$$\epsilon = 4\pi^2 \int_{k_0}^\infty \mathrm{d}k \iint_{\Delta_k} \mathrm{d}p\mathrm{d}q\, k^3 pq\, \widehat{b}(k,p,q) \times \int_0^\infty \mathrm{d}t'\, \check{Q}(k,t')\, \check{Q}(p,t')\, \check{Q}(q,t') \left[\check{Q}(k,0)^{-1} - \check{Q}(p,0)^{-1} \right]. \tag{4.9}$$

These equations permit such similar solutions as

$$\check{Q}(k,t) = \frac{K}{2\pi}\, \epsilon^{2/3}\, k^{-11/3}\, \check{Q}^\dagger(K^{1/2}\epsilon^{1/3}k^{2/3}t)\ , \tag{4.10}$$

with

$$\check{Q}^\dagger(0) = 1\ . \tag{4.11}$$

In terms of $\check{Q}^\dagger$, we can rewrite (4.8) and (4.9) as

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \check{Q}^\dagger(t) = -\int_0^\infty \mathrm{d}p\, J(p) \int_0^t \mathrm{d}t' \check{Q}^\dagger(pt') \tag{4.12}$$

and

$$K^{-3/2} = \int_1^\infty \mathrm{d}k \int_0^1 \mathrm{d}p \int_{\max\{k-p,p\}}^{k+p} \mathrm{d}q\, k^3 pq \int_0^\infty \mathrm{d}t\, \check{Q}^\dagger(k^{2/3}t)\, \check{Q}^\dagger(p^{2/3}t)\, \check{Q}^\dagger(q^{2/3}t) \times \left\{ \left[\widehat{b}(k,p,q) + \widehat{b}(k,q,p) \right] (pq)^{-11/3} - \left[\widehat{b}(t,p,q)\, q^{-11/3} + \widehat{b}(t,q,p)\, p^{-11/3} \right] k^{-11/3} \right\}. \tag{4.13}$$

The energy spectrum is represented, from (2.43) and (4.3), by

$$E(k) = 2\pi k^2\, \check{Q}(k,0) = K\, \epsilon^{2/3}\, k^{-5/3}\ . \tag{4.14}$$

Hence, K is actually the Kolmogorov constant (cf. (2.46)). (In this section for brevity we omit the time argument in $E(k,t)$, which is independent of time.) We solved (4.12) numerically with boundary condition (4.11). The result is shown in figure 1, which is same as figure 1 in Kaneda (1986). Here, $G(k,t,t') = \check{Q}^\dagger(K^{1/2}\epsilon^{1/3}k^{2/3}(t-t'))$. From (4.13) and the above numerical result for $\check{Q}^\dagger$, we can evaluate the Kolmogorov constant to be 1.722 (Kaneda 1986).

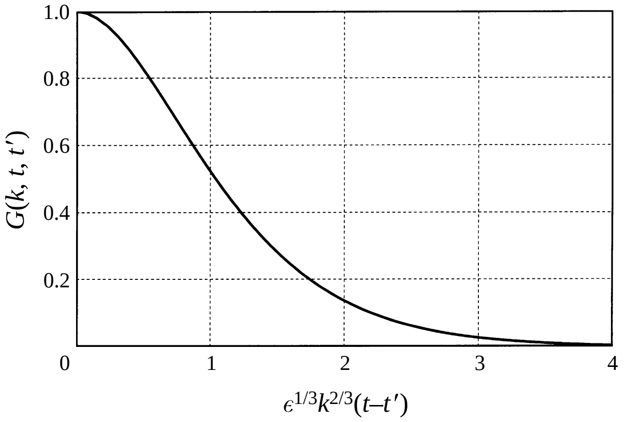


FIGURE 1. Lagrangian velocity response function for stationary turbulence.

4.2. *Energy spectrum*

Here, we solve (4.5) and (4.6) in the entire universal range. We express $\check{Q}$ in terms of non-dimensional functions $\check{Q}^{\ddagger}$ of non-dimensional wavenumber κ and non-dimensional time τ as

$$\check{Q}(k,t) = \frac{1}{2\pi} K\epsilon^{2/3}k^{-11/3}\,\check{Q}^{\ddagger}(\kappa,\tau) \tag{4.15}$$

with

$$\check{Q}^{\ddagger}(0,0) = 1, \tag{4.16}$$

$$\kappa = K^{-3/8}\epsilon^{-1/4}\nu^{3/4}k, \quad \tau = K^{1/2}\epsilon^{1/3}k^{2/3}t\ . \tag{4.17}$$

Then, (4.5) is converted into

$$\frac{\partial^2}{\partial\tau^2}\log\check{Q}^{\ddagger}(\kappa,\tau) = -\int_0^\infty \mathrm{d}p\, J(p)\,\check{Q}^{\ddagger}(\kappa p^{3/2},\tau p) \tag{4.18}$$

with

$$\left.\frac{\partial}{\partial\tau}\log\check{Q}^{\ddagger}(\kappa,\tau)\right|_{\tau=0} = -\kappa^{4/3}\ , \tag{4.19}$$

while (4.6) is written as

$$\check{Q}^{\ddagger}(\kappa,0) = \frac{1}{2}\iint_{\Delta_\kappa}\mathrm{d}p\mathrm{d}q\,(pq)^{-8/3}\kappa^{-1}\,\widehat{b}(\kappa,p,q)$$
$$\times\int_0^\infty \mathrm{d}t'\,\check{Q}^{\ddagger}(\kappa,\kappa^{2/3}t')\,\check{Q}^{\ddagger}(p,p^{2/3}t')\,\check{Q}^{\ddagger}(q,q^{2/3}t')\left[\kappa^{11/3}\,\check{Q}^{\ddagger}(\kappa,0)^{-1} - p^{11/3}\,\check{Q}^{\ddagger}(p,0)^{-1}\right]. \tag{4.20}$$

Note that the energy spectrum is expressed as

$$E(k) = K\epsilon^{2/3}k^{-5/3}\,\check{Q}^{\ddagger}(\kappa,0)\ . \tag{4.21}$$

Hence, by searching numerically (using an iteration method for (4.18) and (4.19), and the Newton–Raphson method for (4.20)) the solution of $\check{Q}^{\ddagger}$ which satisfies (4.18)–(4.20) and (4.16), we can determine the functional form of the energy spectrum through (4.21). The results are shown in figure 2 for (*a*) the three-dimensional energy spectrum, (b) the one-dimensional longitudinal energy spectrum

$$E_{\parallel}(k) = \frac{1}{2}\int_k^\infty \mathrm{d}k' \left(1 - \frac{k^2}{k'^2}\right)\frac{E(k')}{k'} \tag{4.22}$$

and (c) the compensated one-dimensional longitudinal energy spectrum $E_{\parallel}(k)/(\epsilon^{2/3}k^{-5/3})$, which are almost identical to those obtained by a numerical integration of equations (3.16) and (3.22) for decaying turbulence (Gotoh *et al.* 1988). Agreement with measurements of a tidal channel (Grant, Stewart & Moilliet 1962), which plotted by solid and white circles, and other experimental data is excellent.

4.3. *Skewness factor of the velocity derivative*

Although only the second-order moments are dealt with in the present closure theory, the skewness factor of the longitudinal derivative of the velocity, which is a third-order moment, can be calculated with the help of the Kármán-Howarth equation for homogeneous isotropic turbulence as

$$S = \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^3} \Big/ \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}^{3/2} = -\frac{3\sqrt{30}\nu}{7}\int_0^\infty \mathrm{d}k\, k^4\, E(k) \Big/ \left(\int_0^\infty \mathrm{d}k\, k^2\, E(k)\right)^{3/2}. \tag{4.23}$$

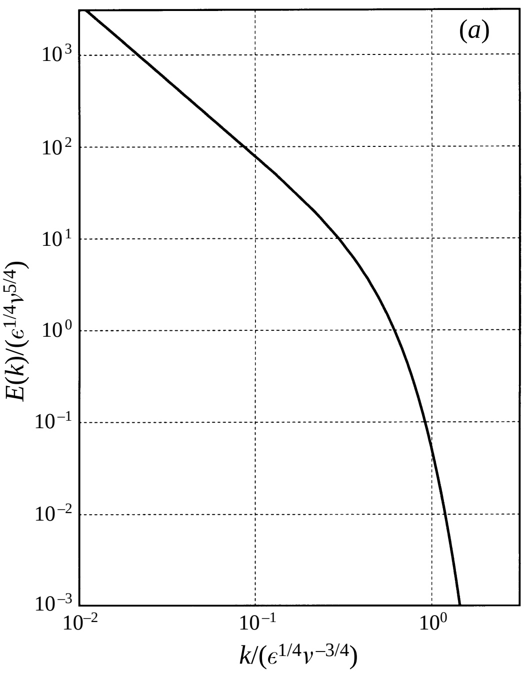
Using the result of the numerical computation for $\check{Q}^{\ddagger}$, we can perform the integrations on the right-hand side of (4.23) to find

$$S = -0.66\,, \tag{4.24}$$

which is in perfect agreement with the value obtained by a numerical integration of the Markovianized LRA equations for the decaying turbulence (Kaneda 1993). This agreement may be attributed to the fact that the structure of the energy spectrum in the universal range is the same for the stationary and the decaying cases (see Appendix F). Note that this factor is independent of Reynolds number. Many turbulence measurements, on the contrary, show that it may increase in magnitude with Reynolds number which expresses the intermittency of turbulence, though it is not conclusive because fluctuations in the data are quite large. It varies from −0.6 to −1 in the range $10^3 < R_\lambda < 2 \times 10^4$ (see Van Atta & Antonia 1980). The present result (4.24) is consistent with observation within this range of the Reynolds number.

4.4. *Eddy viscosity*

One of the main difficulties in analysing the structure of developed turbulence at high Reynolds numbers may be attributed to the enormously wide range of relevant active motions. The ratio between the largest and smallest motions, i.e. the energy-containing and the energy-dissipation scales increases in proportion to the power of 3/4 of the Reynolds number. It is hard to resolve the smallest excited scales of developed turbulence of practical interest even on a present-day supercomputer. The so-called large-eddy simulation (see Lesieur & Métais 1996 for a review) may be one of the most promising methods of analysing the turbulence dynamics in which only large-scale components of motion are explicitly simulated, while the effects on the resolved-scale components of the subgrid-scale motions are implicitly taken into account as eddy viscosity. It is the purpose of the present section to examine the property of the eddy viscosity in the framework of closure theories (Kraichnan 1976).



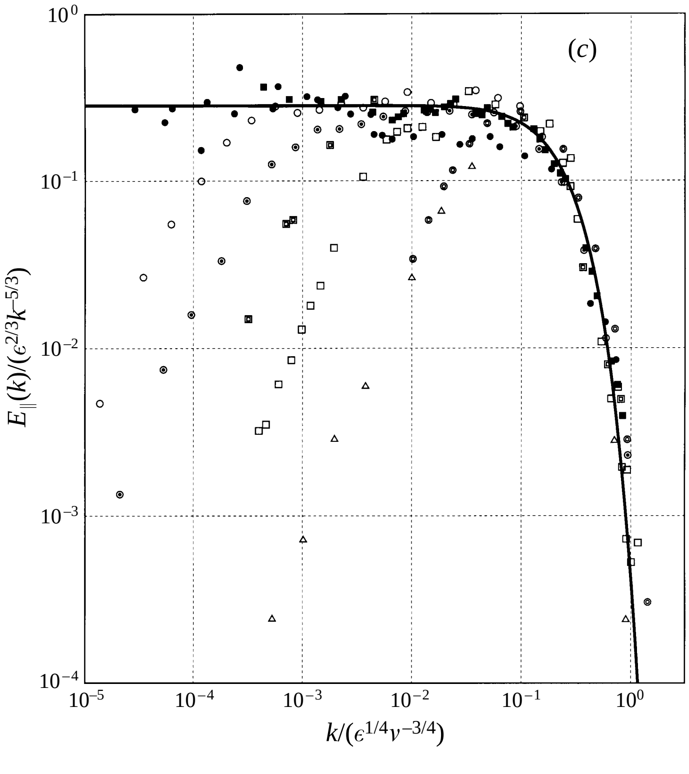


FIGURE 2 (a, c). For caption see facing page.

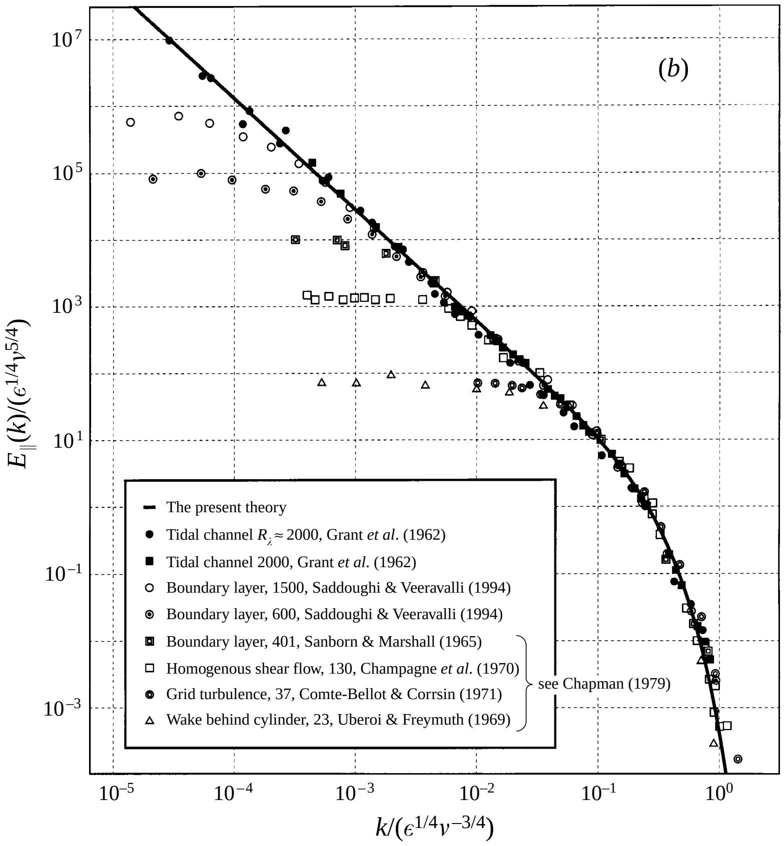


FIGURE 2. Energy spectra in the universal range for stationary turbulence: (*a*) three-dimensional energy spectrum; (*b*) one-dimensional longitudinal energy spectrum; (*c*) one-dimensional compensated longitudinal energy spectrum. Symbols in (*b*) and (*c*) denote experimental data for various kinds of turbulence (taken from Chapman 1979, Grant *et al.* 1962 and Saddoughi & Veeravalli 1994). R_λ is the microscale Reynolds number.

For the discussion of the eddy viscosity it is convenient to rewrite the energy equation (3.22) as

$$\frac{\partial}{\partial t} E(k,t) = -2\nu k^2 E(k,t) + T(k,t) , \tag{4.25}$$

where

$$T(k,t) = \iint_{\triangle_k} \mathrm{d}p\mathrm{d}q \; T^{(3)}(k,p,q,t) , \tag{4.26}$$

and

$$\begin{aligned} T^{(3)}(k,p,q,t) = 2\pi^2 k^3 pq \int_{-\infty}^{t} \mathrm{d}t' \; G(k,t,t')\, G(p,t,t')\, G(q,t,t') \\ \times \Bigg[\left(\widehat{b}(k,p,q) + \widehat{b}(k,q,p) \right) Q(p,t',t')\, Q(p,t',t') \\ - \left(\widehat{b}(k,p,q)\, Q(q,t',t') + \widehat{b}(k,q,p)\, Q(p,t',t') \right) Q(k,t',t') \Bigg] \end{aligned} \tag{4.27}$$

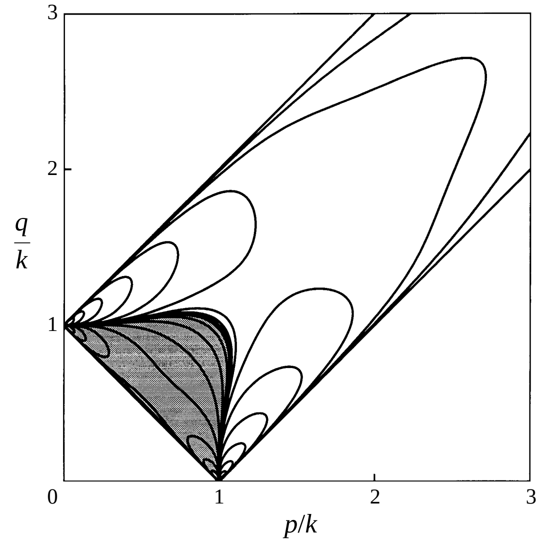


FIGURE 3. Triad energy transfer function $T^{(3)}(k,p,q)/(\epsilon k^3)$ in the inertial range for stationary turbulence. Contour levels are $0, \pm 10^{-1.5}, \pm 10^{-1}, \pm 10^{-0.5} \cdots, \pm 10^2$. Positive regions are shaded.

is the triad energy transfer function. Here, the initial time has been set at $t_0 = -\infty$. The first term on the right-hand side of (4.25) represents the dissipation of energy by molecular viscosity, and the second the energy transfer to the modal energy of wavenumber k from all the other modal energies through nonlinear interactions. Eddy viscosity is defined as the molecular viscosity counterpart when the contribution from the subgrid-scale components is expressed like the first term of (4.25) (see (4.29) below). In figure 3, the triad energy transfer function is shown in the case that all of three wavenumbers, k, p, q are in the inertial range. Positive regions are shaded. Sharp peaks at the corners of the rectangular domain represent strong non-local triad interactions (see Ohkitani & Kida 1992).

Let us denote by k_c the cut-off wavenumber which is the reciprocal of the dividing length of the resolved and the subgrid scales (e.g. the mesh size in a numerical simulation of turbulence). We divide the energy transfer function $T(k,t)$, which is composed of many triad interactions, into two parts as

$$T(k,t) = T^{<}(k,t|k_c) + T^{>}(k,t|k_c), \tag{4.28}$$

where $T^{<}$ denotes the contribution from the resolved scales, i.e. an integral of (4.26) over p and $q \leqslant k_c$, and $T^{>}$ that from the subgrid scales, i.e. an integral for p or $q > k_c$. If we write the second term of (4.28) formally as

$$T^{>}(k,t|k_c) = -2k^2\, v_T(k,t|k_c) E(k,t)\ , \tag{4.29}$$

then $v_T(k,t|k_c)$ is regarded as the eddy viscosity since it is a molecular viscosity counterpart (cf. the first term of (4.25)). Notice that the eddy viscosity varies depending upon the relevant wavenumber, contrary to the molecular viscosity. In the following we will estimate the wavenumber dependence of the eddy viscosity for both wavenumbers k and k_c lying in the inertial range of stationary turbulence.

On substitution of (4.10) and (4.14) in (4.29), we obtain

$$v_T(k,t|k_c) = K^{1/2}\, \epsilon^{1/3}\, k_c^{-4/3}\, I(k/k_c) \tag{4.30}$$

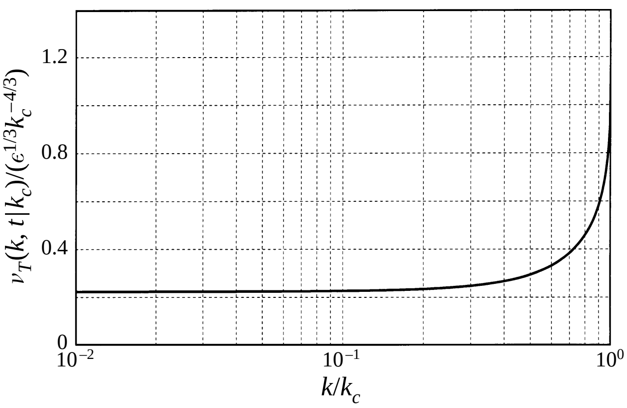


FIGURE 4. Wavenumber dependence of eddyviscosity. It is nearly constant $\nu_T \approx 0.224\epsilon^{1/3}k_c^{-4/3}$ at small wavenumbers ($k \ll k_c$). The variation is less than 50% up to $k \approx 0.6k_c$. At cut-off wavenumber k_c, it is 5.7 times as large as that at small wavenumbers.

with

$$I(k/k_c) = -\tfrac{1}{3}\,(k/k_c)^{-4/3} \int_0^\infty \mathrm{d}t \int_{(k/k_c)^{-1}t}^\infty \mathrm{d}p \int_{p-t}^{p} \mathrm{d}q\; t^3 pq\, \check{Q}^\dagger(t^{2/3})\, \check{Q}^\dagger(p^{2/3})\, \check{Q}^\dagger(q^{2/3}) \times \left[(pq)^{-11/3} \left(\widehat{b}(t,p,q) + \widehat{b}(t,q,p) \right) - (tp)^{-11/3}\, \widehat{b}(t,q,p) - (tq)^{-11/3}\, \widehat{b}(t,p,q) \right]. \tag{4.31}$$

Integration of (4.31) is carried out using the numerical solution of $\check{Q}^\dagger$ already obtained in §4.1. At two extreme values of k/k_c we find $I(0) = 0.170$ and $I(1) = 0.970$. The eddy viscosity thus determined is shown in figure 4. At wavenumbers much smaller than the cut-off wavenumber ($k \ll k_c$) the eddy viscosity is nearly constant, $\nu_T \approx 0.224\epsilon^{1/3}k_c^{-4/3}$ (Kaneda 1986) . The variation is less than 50% up to $k \approx 0.6k_c$. However, as the wavenumber concerned approaches the cut-off wavenumber, the eddy viscosity increases more and more rapidly. At the cut-off wavenumber it is 5.7 times as large as that at small wavenumbers. This sharp increase of the eddy viscosity near the cut-off wavenumber, which is caused by strong non-local triad interactions such as $p \ll k \approx q$ or $q \ll k \approx p$ (see figure 3), is also observed in other closure theories (Kraichnan 1976), large-eddy simulations (Lesieur & Rogallo 1989), and direct numerical simulations (Domaradzki, Liu & Brachet 1993).

5. Decaying turbulence

In this section, we consider homogeneous isotropic freely decaying turbulence with the set of equations (3.16) and (3.22). The initial value problem of these equations was investigated numerically by Gotoh *et. al* (1988). Here, instead, we seek solutions to these equations in a similarity form. It can be shown that there are in general no similar solutions with a single similarity law that hold over the entire wavenumber range. Therefore, as was done by one of the present authors for the modified zero-fourth-order cumulant approximation (Kida 1981), we seek similar solutions which obey different similarity laws in two wavenumber ranges, namely they are characterized by total energy $\mathscr{E}(t)$ and energy dissipation rate $\epsilon(t)$ in the energy-containing range, and by $\epsilon(t)$ and ν in the universal range.

5.1. *Similarity form in the energy-containing range*

It is easy to show that the similarity form of equations (3.16) and (3.22) in the universal range is the same whether for decaying or stationary turbulence (see Appendix F). That in the energy-containing range, on the other hand, is derived as follows. We start by introducing index ζ which characterizes large-scale structure by

$$E(k,t) \propto k^{\zeta} \qquad \text{as} \quad k \to 0 \; . \tag{5.1}$$

If we require that the velocity correlation tensor $\widetilde{V}_{ij}$ does not diverge at the origin, then ζ is limited from below to 2. On the other hand, even if ζ is greater than 4 at the initial instant, ζ changes immediately to 4 (see (5.9) below). We restrict, therefore, ourselves to the range

$$2 \leqslant \zeta \leqslant 4 \; . \tag{5.2}$$

It is convenient for the following analysis to introduce a new variable $\widehat{E}_{\zeta}(k,t)$ by

$$E(k,t) = \widehat{E}_{\zeta}(k,t)\, k^{\zeta} \; , \tag{5.3}$$

where

$$0 < \widehat{E}_{\zeta}(0,t) < \infty \; . \tag{5.4}$$

Then, (2.43) and (3.22) lead to

$$\left[\frac{\partial}{\partial t} + 2\nu k^2 \right] \widehat{E}_{\zeta}(k,t) = \iint_{\triangle_k} \mathrm{d}p\mathrm{d}q \; k^{3-\zeta} p q^{\zeta-1}\, \widehat{b}(k,p,q)$$
$$\times \int_0^t \mathrm{d}t' \; G(k,t,t')\, G(p,t,t')\, G(q,t,t')\, \widehat{E}_{\zeta}(q,t') \left[p^{\zeta-2}\, \widehat{E}_{\zeta}(p,t') - k^{\zeta-2}\, \widehat{E}_{\zeta}(k,t') \right], \tag{5.5}$$

where we have put $t_0 = 0$. Equation (3.16) for the response function is rewritten as

$$\left[\frac{\partial}{\partial t} + 2\nu k^2 + \tfrac{2}{3} k^{\zeta+3} \int_0^{\infty} \mathrm{d}p\, p^{4/3+\zeta}\, J(p^{2/3}) \int_{t'}^{t} \mathrm{d}t'' \; \widehat{E}_{\zeta}(kp,t'')\, G(kp,t,t'') \right] G(k,t,t') = 0. \tag{5.6}$$

If we demand that the turbulence in the energy-containing range is characterized only by k, $\mathscr{E}(t)$ and $\epsilon(t)$, functions $\widehat{E}_{\zeta}$ and G may be written, from the dimensional analysis, as

$$\widehat{E}_{\zeta}(k,t) = \mathscr{E}(t)^{(3\zeta+5)/2}\, \epsilon(t)^{-\zeta-1}\, E_{\zeta}^{\dagger}\Big(\mathscr{A}\mathscr{E}(t)^{3/2}\, \epsilon(t)^{-1}\, k \Big), \tag{5.7}$$

$$G(k,t,t') = G^{\dagger}\Big(\mathscr{A}\mathscr{E}(t)^{3/2}\, \epsilon(t)^{-1}\, k \, , \, \mathscr{A}\mathscr{E}(t')^{3/2}\, \epsilon(t')^{-1}\, k \Big), \tag{5.8}$$

where $\mathscr{A}$ is a non-dimensional constant, which will be determined later so that the final expression may be simple (see (5.19) below). By substituting these similarity forms into (5.5) and (5.6), we find that the viscous term is smaller than the time-derivative and the nonlinear terms by a factor Re^{-1}, where Re is the Reynolds number ($Re = \mathscr{E}^2/\epsilon\nu$). Hence, in the limit of large Reynolds numbers, the viscous terms can be neglected.

By neglecting the viscous term and taking the limit of $k \to 0$ in (5.5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\widehat{E}_\zeta(0,t) = \begin{cases} 0 & (2 \leqslant \zeta < 4)\ , \quad (5.9a) \\ \dfrac{14}{15}\displaystyle\int_0^\infty \mathrm{d}p \int_0^t \mathrm{d}t' \left[p^3\, G(p,t,t')\, \widehat{E}_4(p,t') \right]^2 & (\zeta = 4)\ . \quad (5.9b) \end{cases}$$

This equation tells us that the Birkhoff constant, which is $\widehat{E}_2(0,t)$, is invariant in time, but the Loitsiansky integral, which is equal to $\widehat{E}_4(0,t)$, is not, as pointed out before in other closure theories of turbulence (Lesieur & Schertzer 1978; Kida 1981). Substitution of similarity form (5.7) into (5.9a) leads to $\mathscr{E}(t)$ and $\epsilon(t)$ having power functions of t as

$$\mathscr{E}(t) = \mathscr{E}_0\, t^{-\sigma}\ , \tag{5.10}$$

where

$$\sigma = \frac{2(\zeta+1)}{\zeta+3} \qquad (2 \leqslant \zeta < 4)\ . \tag{5.11}$$

Here, we have used relation

$$\epsilon(t) = -\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t}\ . \tag{5.12}$$

For $\zeta = 4$, on the other hand, it may not be possible to prove that $\mathscr{E}(t)$ is a power function of t. However, if we assume (5.10), then σ may be evaluated by

$$\frac{(2-\sigma)(10-7\sigma)}{2\,\sigma^2}\, E_4^\dagger(0) = \frac{28}{15}\,\mathscr{A}^{-7} \int_0^\infty \mathrm{d}p \int_0^1 \mathrm{d}t t^{(20-13\sigma)/(2-\sigma)} \left[p^3\ G^\dagger(p,pt)\, E_4^\dagger(pt) \right]^2 \qquad (\zeta = 4). \tag{5.13}$$

Inversely, it is easy to show that (5.10), (5.11) and (5.13) are sufficient conditions for the existence of similar solutions (5.7) and (5.8).

By making use of (5.10) and (5.12), we can rewrite (5.7) and (5.8) as

$$\widehat{E}_\zeta(k,t) = \mathscr{E}_0^{(\zeta+3)/2}\, \sigma^{-\zeta-1}\, t^{(-\sigma\zeta-3\sigma+2\zeta+2)/2}\, E_\zeta^\dagger\Big(\mathscr{A}\, \mathscr{E}_0^{1/2}\, \sigma^{-1}\, t^{-\sigma/2+1} k \Big), \tag{5.14}$$

$$G(k,t,t') = G^\dagger\Big(\mathscr{A}\, \mathscr{E}_0^{1/2}\, \sigma^{-1}\, t^{-\frac{\sigma}{2}+1} k\, ,\, \mathscr{A}\, \mathscr{E}_0^{1/2}\, \sigma^{-1}\, t'^{-\frac{\sigma}{2}+1} k \Big). \tag{5.15}$$

In order to make the final equations simpler, we further replace $E^\dagger$, $G^\dagger$ and σ with

$$E_\zeta^\dagger(x) = \mathscr{A}^{\zeta+3} \left(\frac{b}{2-3b} \right)^2 x^{-\zeta-5/3}\, E^\ddagger(x^{2/3})\ , \tag{5.16}$$

$$G^\dagger(x,x') = G^\ddagger(x^{2/3}, x'^{2/3}) \tag{5.17}$$

and

$$\sigma = 2 - 3b\ , \tag{5.18}$$

respectively. If we choose

$$\mathscr{A} = \frac{2-3b}{\mathscr{E}_0^{1/2}\, b^{3/2}}\ , \tag{5.19}$$

then (5.14) and (5.15) are rewritten as

$$\widehat{E}_\zeta(k,t) = t^{2(b-1)}\, k^{-\zeta-5/3}\, E^\ddagger(b^{-1} t^b k^{2/3})\ , \tag{5.20}$$

$$G(k,t,t') = G^\ddagger(b^{-1} t^b k^{2/3}, b^{-1} t'^b k^{2/3})\ , \tag{5.21}$$

and (5.5) and (5.6) as

$$\frac{\partial}{\partial t}\left(t^{2-2/b}\, E^{\ddagger}(t) \right) = t^{\frac{1}{b}-1} \iint_{\Delta_1} \mathrm{d}p\mathrm{d}q\; pq\, \widehat{b}(1,p,q) \int_0^t \mathrm{d}t'\; t'^{3-3/b}$$
$$\times\, G^{\ddagger}(t,t')\, G^{\ddagger}(p^{2/3}t, p^{2/3}t')\, G^{\ddagger}(q^{2/3}t, q^{2/3}t')\, q^{-11/3}\, E^{\ddagger}(q^{-11/3}t')$$
$$\times \left[\, q^{-11/3}\, E^{\ddagger}(p^{-11/3}t') - E^{\ddagger}(t')\, \right] \tag{5.22}$$

and

$$\frac{\partial}{\partial t}\, \log G^{\ddagger}(t,t') = -t^{1/b-1} \int_0^\infty \mathrm{d}pJ(p) \int_{t'}^{t} \mathrm{d}t''\; t''^{1-1/b}\, E^{\ddagger}(pt'')\, G^{\ddagger}(pt,pt'')\;, \tag{5.23}$$

respectively. Conditions (5.11) and (5.13) for σ are also rewritten as

$$b = \begin{cases} \dfrac{4}{3(\zeta+3)} \quad (2 \leqslant \zeta < 4), & (5.24a) \\ \left[\dfrac{21}{4} - \dfrac{7}{10\, E_0^{\ddagger}} \displaystyle\int_0^\infty \mathrm{d}k\, k^{-15/2} \int_0^1 \mathrm{d}t\, t^{3-3/b} \left(G^{\dagger}(k,kt)\, Q^{\dagger}(kt) \right)^2 \right]^{-1} (\zeta = 4). & (5.24b) \end{cases}$$

Here, $E_0^{\ddagger}$ is defined by

$$E^{\ddagger}(x) = E_0^{\ddagger}\, x^{\zeta+5/3} \qquad \text{as} \quad x \to 0\;, \tag{5.25}$$

which follows from (5.4) and (5.20).

Finally, we consider the boundary conditions for $E^{\ddagger}$ and $G^{\ddagger}$. Energy spectrum $E(k,t)$ is expressed, from (5.3) and (5.20), as

$$E(k,t) = t^{2(b-1)}\, k^{-5/3}\, E^{\ddagger}(b^{-1}t^b k^{2/3})\;. \tag{5.26}$$

If we demand that $E(k,t)$ at large wavenumbers in the inertial range be connected smoothly with the $k^{-5/3}$ spectrum which is realized at lower wavenumbers in the universal range, then $E^{\ddagger}$ must approach a constant, which we can choose, without loss of generality, as unity, i.e.

$$E^{\ddagger}(\infty) = 1\;. \tag{5.27}$$

Then, we have

$$E(k,t) = k^{-5/3} t^{-2+2b} = \left(\mathscr{E}_0(2-3b)\right)^{-2/3} \epsilon(t)^{2/3} k^{-5/3} \left(= K\, \epsilon(t)^{2/3} k^{-5/3},\ \text{say}\right) \quad \text{as } k \to \infty, \tag{5.28}$$

where use has been made of (5.10), (5.12), (5.18) and (5.26). Integration of (5.26) with respect to k together with $\mathscr{E}(t) = \int_0^\infty \mathrm{d}kE(k,t)$ gives

$$\mathscr{E}_0 = \frac{3}{2b} \int_0^\infty \mathrm{d}x\, x^{-2}\, E^{\ddagger}(x)\;. \tag{5.29}$$

Hence, the Kolmogorov constant is represented, from (5.28) and (5.29), as

$$K = \left[\frac{3}{2} \left(\frac{2}{b} - 3 \right) \int_0^\infty \mathrm{d}x\, x^{-2}\, E^{\ddagger}(x) \right]^{-2/3}\;. \tag{5.30}$$

A boundary condition for $G^{\ddagger}$ follows from initial condition (3.17) as

$$G^{\ddagger}(x,x) = 1\;. \tag{5.31}$$

Thus, we have obtained a system of integro-differential equations (5.22)–(5.24) to be solved with boundary conditions (5.27) and (5.31).

5.2. *Kolmogorov constant*

In the limit that $t' \to \infty$, equation (5.23) of the response function becomes

$$\frac{\partial}{\partial t} \log G^{\ddagger}(t,t') = -\int_0^{\infty} \mathrm{d}p \, J(p) \int_{t'}^{t} \mathrm{d}t'' \, G^{\ddagger}(pt, pt'') \quad \text{as } t' \to \infty \, , \tag{5.32}$$

where use has been made of

$$\int_{t'}^{t} \mathrm{d}t'' \, t''^{1-1/b} \, E^{\ddagger}(pt'') \, G^{\ddagger}(pt,pt'') \, = t^{1-1/b} \int_{t'}^{t} \mathrm{d}t'' \, G^{\ddagger}(pt,pt'') \quad \text{as } t' \to \infty \, . \tag{5.33}$$

Equation (5.32) with (5.31) permits a solution such as

$$G^{\ddagger}(t,t') = G^{\ddagger}_{\infty}(t-t') \, , \tag{5.34}$$

where $G^{\ddagger}_{\infty}$ obeys

$$\frac{\mathrm{d}}{\mathrm{d}t} \log G^{\ddagger}_{\infty}(t) = -\int_0^{\infty} \mathrm{d}p \, J(p) \int_0^{t} \mathrm{d}t' \, G^{\ddagger}_{\infty}(pt') \, , \tag{5.35}$$

with boundary condition

$$G^{\ddagger}_{\infty}(0) = 1 \, . \tag{5.36}$$

Notice that (5.35) is identical to (4.12). The functional form of $G^{\ddagger}_{\infty}$ does, therefore, coincide with that of $\check{Q}^{\dagger}$, which is the response function for the stationary case (figure 1)†.

The energy equation (5.22), on the other hand, is reduced to (see Appendix G for the derivation)

$$\begin{aligned} &\frac{3}{2}\left(\frac{2}{b}-3\right) \int_0^{\infty} \mathrm{d}x \, x^{-2} \, E^{\ddagger}(x) \\ &= \int_1^{\infty} \mathrm{d}t \int_0^1 \mathrm{d}p \int_{\max\{t-p,p\}}^{t+p} \mathrm{d}q \, t^3 pq \int_0^{\infty} \mathrm{d}t' \, G^{\ddagger}_{\infty}(t^{2/3}t') \, G^{\ddagger}_{\infty}(p^{2/3}t') \, G^{\ddagger}_{\infty}(q^{2/3}t') \\ &\times \left\{ \left[\widehat{b}(t,p,q) + \widehat{b}(t,q,p)\right] (pq)^{-11/3} - \left[\widehat{b}(t,p,q) \, q^{-11/3} + \widehat{b}(t,q,p) \, p^{-11/3}\right] t^{-11/3} \right\}. \end{aligned} \tag{5.37}$$

Hence, it follows from (5.30) and (5.37) that the Kolmogorov constant K is

† Here, note the difference in the arguments of $\check{Q}^{\dagger}$ and $G^{\ddagger}_{\infty}$. The former is $K^{1/2}\epsilon^{1/3}k^{2/3}(t-t')$, while, the latter is $K^{1/2}k^{2/3}(\epsilon(t)^{1/3}t - \epsilon(t')^{1/3}t')/b$ (see (4.10) and the first equation in §5.3). However, these two agree with each other in the limit of $t' \to \infty$ because

$$K^{1/2}k^{2/3}(\epsilon(t)^{1/3}t - \epsilon(t')^{1/3}t')/b = k^{2/3}(t^b - t'^b)/b \, ,$$

which is shown from (5.10), (5.12), (5.18) and (5.28).

represented as

$$K^{-3/2} = \int_1^\infty \mathrm{d}t \int_0^1 \mathrm{d}p \int_{\max\{t-p,p\}}^{t+p} \mathrm{d}q\, t^3 pq \int_0^\infty \mathrm{d}t'\, G_\infty^\ddagger(t^{2/3}t')\, G_\infty^\ddagger(p^{2/3}t')\, G_\infty^\ddagger(q^{2/3}t') \times \left\{ \left[\widehat{b}(t,p,q) + \widehat{b}(t,q,p)\right](pq)^{-11/3} - \left[\widehat{b}(t,p,q)\, q^{-11/3} + \widehat{b}(t,q,p)\, p^{-11/3}\right] t^{-11/3} \right\}. \tag{5.38}$$

By remembering that $G_\infty^\ddagger$ is identical to $\check{Q}^\dagger$ and comparing (4.13) with (5.38), we can conclude that the Kolmogorov constant in decaying turbulence is same as that in stationary turbulence, i.e. $K = 1.722$.

5.3. *Two-similarity-range solution*

Equations (5.22)–(5.24) for the energy-containing range are solved numerically under boundary conditions (5.27) and (5.31) for two extreme cases of index ζ, i.e. $\zeta = 2$ and 4. In the case of $\zeta = 2$, parameter b is fixed by (5.24*a*) as $b = \frac{4}{15}$. Equations (5.22) and (5.23) are then solved iteratively as described in Appendix H. The response function and the energy spectrum function thus obtained are plotted in figures 5(*a*) and (*b*), respectively. The response function and the energy spectrum function are respectively written in terms of solutions of (5.22) and (5.23) as

$$G(k,t,t') = G^\ddagger(\tau/b, \tau'/b),$$

$$E(k,t)\big/(\mathscr{E}^{5/2}\epsilon^{-1}) = K\,(\mathscr{E}^{3/2}\epsilon^{-1}k)^{-5/3}\, E^\ddagger(K^{1/2}(2b^{-1}-3)(\mathscr{E}^{3/2}\epsilon^{-1}k)^{2/3}),$$

where use has been made of (4.17), (5.10), (5.12), (5.18), (5.21), (5.26) and (5.28). Here $\tau' = K^{1/2}\epsilon(t')^{1/3}k^{2/3}t'$. The almost equi-distance of the contours of logarithmic levels in figure 5(*a*) indicates that the response function decays exponentially with response time $\tau - \tau'$. The characteristic decay time of the response function takes a non-zero finite value at initial time $\tau' = 0$ and decreases with time τ' in this non-dimensional response time $\tau - \tau'$. In the original physical time t, however, it is a monotonically increasing function of time starting from zero at the initial instant (figures are omitted). The energy spectrum, shown in figure 5(*b*), has asymptotic forms, $\propto k^2$ and $\propto k^{-5/3}$, at small and large wavenumbers respectively as imposed as the boundary conditions.

In the case of $\zeta = 4$, on the other hand, parameter b is not known *a priori* but must be determined iteratively together with $E^\ddagger$ and $G^\ddagger$ (cf. (5.24*b*)). We obtained numerically that

$$b = 0.207 \qquad \text{for} \quad \zeta = 4\,, \tag{5.39}$$

which gives, through (5.18), the power exponent of the energy

$$\sigma = 1.38 \qquad \text{for} \quad \zeta = 4\,. \tag{5.40}$$

Interestingly, this value is exactly the same as the one predicted by the EDQNM theory (Lesieur & Schertzer 1978) as well as by the modified zero-fourth-order cumulant approximation (Kida 1981). As shown in figures 6(*a*) and 6(*b*), the shape of the response function and the energy spectrum function are qualitatively the same as for $\zeta = 2$.

As mentioned in the beginning of §5, there are in general no overall similarity solutions with a single similarity law valid over the entire wavenumber range. Instead,

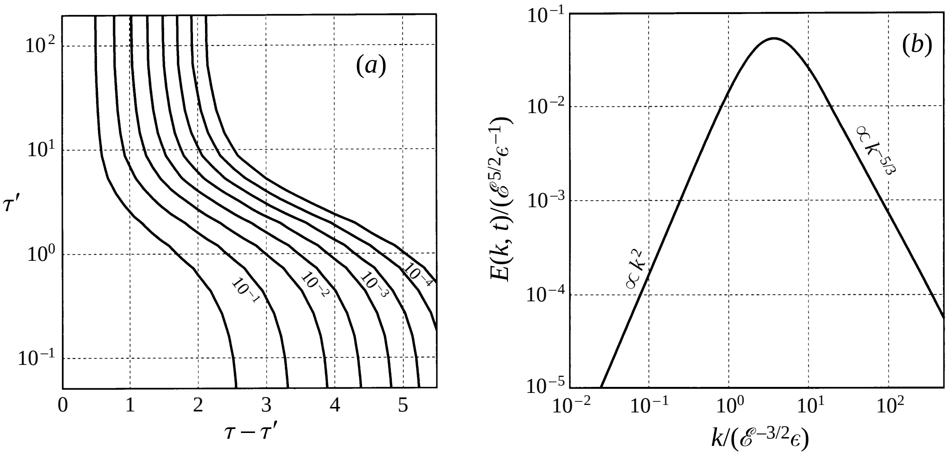


FIGURE 5. Similar solutions in the energy-containing and the inertial ranges of freely decaying turbulence for $\zeta = 2$. (a) Lagrangian velocity response function $G(k, t, t')$. Here, $\tau = K^{1/2}\epsilon(t)^{1/3}k^{2/3}t$ and $\tau' = K^{1/2}\epsilon(t')^{1/3}k^{2/3}t'$. Contour levels are 10^x $(x = -0.5, -1, -1.5, \cdots, -4)$. ($b$) Three-dimensional energy spectrum function.

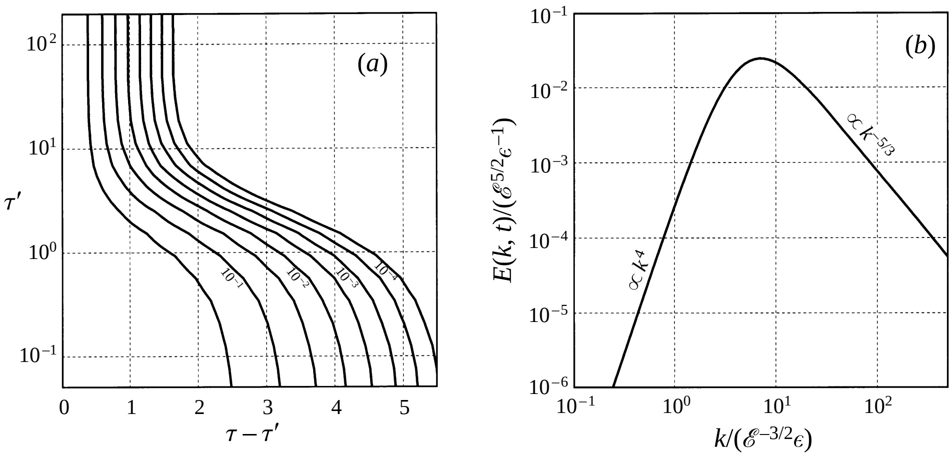


FIGURE 6. Same as figure 5 but for $\zeta = 4$.

the energy spectrum obeys different similarity laws in the energy-containing range and in the universal range, and it is connected smoothly between them. It follows from (5.10) and (5.12) that the normalized energy and wavenumber depend on time, in the respective wavenumber ranges, as

$$\frac{E(k,t)}{\mathscr{E}^{5/2}\epsilon^{-1}} \propto t^{3\sigma/2-1}, \qquad \frac{k}{\mathscr{E}^{-3/2}\epsilon} \propto t^{-\sigma/2+1} \qquad \text{(in the energy-containing range)}\,, \quad (5.41a)$$

$$\frac{E(k,t)}{\epsilon^{1/4}\nu^{5/4}} \propto t^{(\sigma+1)/4}, \qquad \frac{k}{\epsilon^{1/4}\nu^{-1/4}} \propto t^{(\sigma+1)/4} \qquad \text{(in the universal range)}\,. \quad (5.41b)$$

Note that the two similarity laws coincide with each other only for $\sigma = 1$ for which the energy spectral density diverges at zero wavenumber, or the three-dimensional energy spectrum behaves as $E(k, t) \propto k$ (as $k \to 0$). In this case, the total energy decreases in inverse proportion to time, which has been observed often in grid-generated turbulence (see Batchelor 1953).

The time evolution of the energy spectrum with two similarity decay laws is depicted in the entire wavenumber range for cases $\zeta = 2$ (figure 7*a*) and 4 (figure 7*b*). In each figure, two inserted panels (which are identical to figures 5(*b*) and 2(*a*) for $\zeta = 2$, and 6(*b*) and 2(*a*) for $\zeta = 4$) represent respectively the energy-containing and the universal ranges, which translate in time in the directions indicated by arrows. Notice that the direction of the arrow in the energy-containing range is exactly parallel to the asymptotic slope at the small wavenumber of the energy spectrum for case $\zeta = 2$, which exhibits the invariance of the Birkhoff constant, whereas they are slightly inclined to each other in the case $\zeta = 4$, which implies that the invariance of the Loitsiansky integral is slightly broken (Batchelor & Proudman 1956).

6. Concluding remarks

One of the most important (at least from a fundamental theoretical point of view) and unsolved problems in the analytical theories of turbulence is to deduce (a part of) the statistical properties that turbulence might have from the basic equations, i.e. the Navier–Stokes equation. Complete information on the statistics is included in Hopf's (1952) functional formulation of the probability distribution function of velocity, which is unfortunately formidable to solve. On the other hand, if we restrict our interest to a few lower-order moments of velocity which are practically more important, such as the mean velocity distribution, the velocity correlation function, and try to construct evolution equations for these moments, then we encounter a closure problem originating from the nonlinearity in the Navier–Stokes equation. This problem has long been preventing us from constructing any rigorous (in the limit of large Reynolds numbers) theories despite much effort by many researchers. There are quite a few phenomenological theories which predict practically useful results. Among others, the EDQNM theory, the k-ϵ and the Reynolds stress models may be named as most successful ones. Notice that all of these theories have one or more free parameters to be adjusted for a better agreement with observation.

The current work is one attempt to construct an analytical theory without any adjustable parameters. The closed system of equations for the Lagrangian velocity correlation and the response functions derived in this paper by the direct-interaction approximation (Kraichnan 1959) is exactly same as those equations that were derived before by the reversed expansion method (called LRA by Kaneda 1981). It may be a hint of the robustness of this theory that two different approaches lead to the same results. Strictly speaking, the present theory cannot be said to be a rational approximation because it is based upon several unproved (but intuitively reasonable) assumptions and procedures summarized in §3.2. Nevertheless, the predicted energy spectrum for homogeneous isotropic turbulence agrees excellently with many measurements of real turbulence over the whole universal range without any adjustable parameters. It is therefore very interesting to check the validity of the basic assumptions, e.g. by the direct numerical simulation of the Navier–Stokes equation.

Several useful findings obtained by the present theory are summarized as follows. We have proved within the current closure theory that the form of the energy spectrum in the universal range is common between a stationary and a decaying turbulence if we appropriately normalize the energy spectrum and the wavenumber in terms of the time-dependent energy dissipation rate $\epsilon(t)$. Two similarity-range solutions for decaying turbulence may serve as reference spectra to be compared with experiments

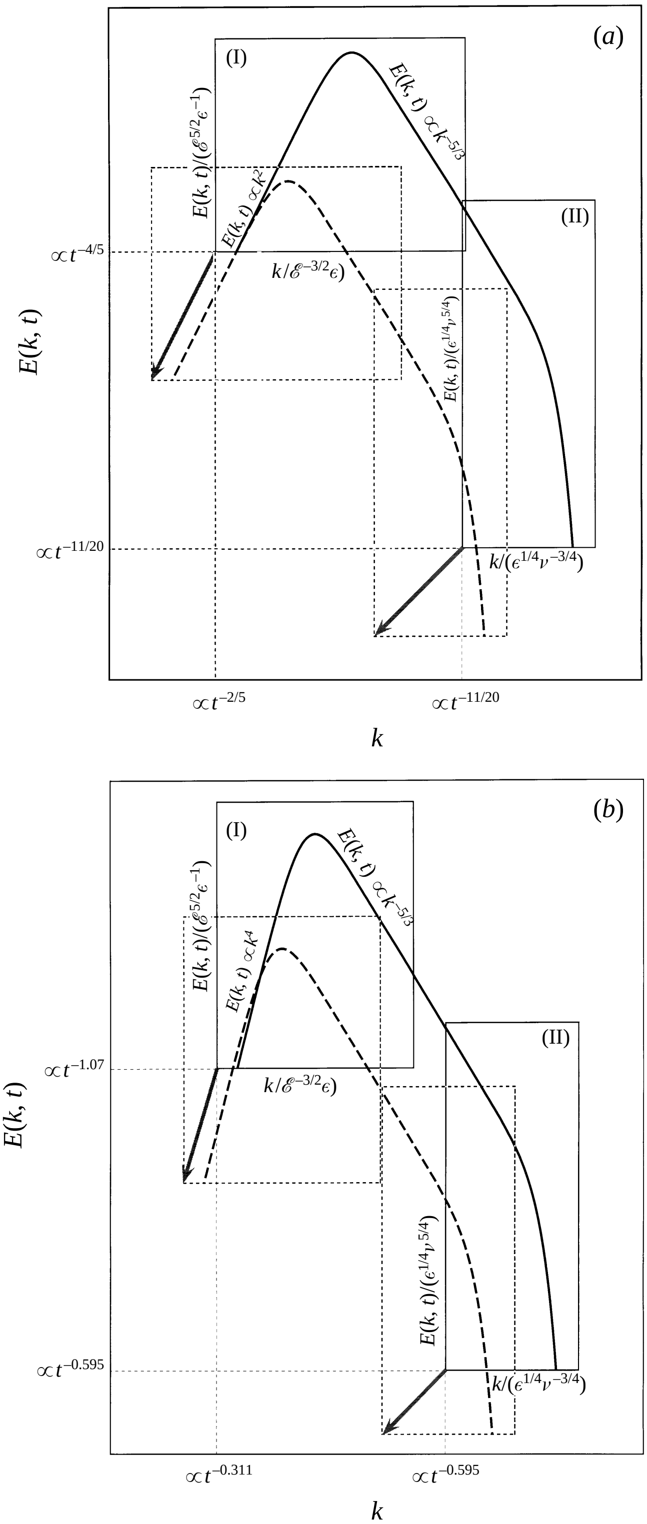


FIGURE 7. Time evolution of the three-dimensional energy spectrum of freely decaying turbulence with two similarity laws for (*a*) $\zeta = 2$ and (*b*) $\zeta = 4$. The two inserted panels represent (I) the energy-containing and the inertial ranges, and (II) the universal range, respectively. The energy spectra in these two ranges are connected smoothly in the inertial range between them. The two ranges move in this double logarithmic scale in the direction indicated by arrows according to the respective similarity laws.

if accurate measurements of decaying turbulence becomes possible in the future. The wavenumber dependence of the eddy viscosity, which is often used in large-eddy simulation of turbulence, is estimated to give comparable results with other closure theories, such as the test field model (Kraichnan 1976). The skewness factor of the velocity derivative is obtained to be equal to −0.66 and is independent of the Reynolds number. This value is consistent with available experimental data in the Reynolds number range $10^3 < R_\lambda < 2 \times 10^4$ though it is not known whether there will appear a significant difference at larger Reynolds numbers. Incidentally, effects of intermittency of turbulence fluctuations seem to be irrelevant to the lower-order (e.g. second and third order) moments of velocity considered in the present theory.

The resultant integro-differential equations in the present closure theory look more involved than the original Navier–Stokes equations. Since, however, the averaged quantities, which are expected to be smooth both in space and in time, are employed in the statistical theory, we can take larger mesh sizes both in space and in time in the statistical theory than in the direct numerical simulation which solves an individual velocity field of much more complicated spatio-temporal structure. An application to anisotropic turbulence as well as inhomogeneous turbulence may be tractable next targets.

This paper is dedicated to Professor K. Gotoh on the 65th anniversary of his birth. One of the authors (S.G.) would like to thank Professor S. Toh for his encouragement during this study. We are grateful to Dr S. G. Saddoughi for providing us with their raw data of the energy spectrum plotted in figure 2(b,c). This work was partially supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture.

Appendix A

In §3.1, we introduced a direct-interaction decomposition for the Eulerian velocity field and constructed the governing equations for its NDI-field and DI-field. Here, we will adopt it for the Eulerian velocity response function, the position function and the position response function, i.e.

$$\widetilde{G}^{(E)}_{ij}(\boldsymbol{k},t|\boldsymbol{k}',t') = \widetilde{G}^{(E0)}_{ij}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0) + \widetilde{G}^{(E1)}_{ij}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\ , \qquad (\mathrm{A}\,1)$$

$$\widetilde{\psi}(\boldsymbol{k},t|\boldsymbol{k}',t') = \widetilde{\psi}^{(0)}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0) + \widetilde{\psi}^{(1)}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0) \qquad (\mathrm{A}\,2)$$

and

$$\widetilde{\Psi}_i(\boldsymbol{k},t|\boldsymbol{k}',\boldsymbol{k}'',t') = \widetilde{\Psi}^{(0)}_i(\boldsymbol{k},t|\boldsymbol{k}',\boldsymbol{k}'',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0) + \widetilde{\Psi}^{(1)}_i(\boldsymbol{k},t|\boldsymbol{k}',\boldsymbol{k}'',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\ . \qquad (\mathrm{A}\,3)$$

Now, we construct the governing equations for the six quantities defined by the above three decompositions. First, we can show that, from (2.31) and (2.32), $\widetilde{G}^{(E0)}_{ij}$ obeys

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] \widetilde{G}^{(E0)}_{ij}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)$$
$$= -\mathrm{i}\left(\frac{2\pi}{L}\right)^3 \widetilde{P}_{imn}(\boldsymbol{k}) \sum_{\boldsymbol{p}} {\sum_{\boldsymbol{q}}}' _{(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})} \widetilde{u}_m(-\boldsymbol{p},t)\, \widetilde{G}^{(E0)}_{nj}(-\boldsymbol{q},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0) \qquad (\mathrm{A}\,4)$$

with initial condition

$$\widetilde{G}^{(E0)}_{ij}(\boldsymbol{k},t'|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0) = \frac{L^3}{(2\pi)^6}\,\delta_{ij}\,\delta^3_{\boldsymbol{k}+\boldsymbol{k}'}\ , \tag{A 5}$$

while $\widetilde{G}^{(E1)}_{ij}$ obeys

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + \nu k^2\right]\widetilde{G}^{(E1)}_{ij}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\
&\quad = -\mathrm{i}\left(\frac{2\pi}{L}\right)^3 \widetilde{P}_{imn}(\boldsymbol{k}) \sum_{\substack{\boldsymbol{p}\quad \boldsymbol{q}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\!\!{\sum}' \widetilde{u}_m(-\boldsymbol{p},t)\,\widetilde{G}^{(E1)}_{nj}(-\boldsymbol{q},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\
&\qquad -\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0}\,\mathrm{i}\left(\frac{2\pi}{L}\right)^3 \widetilde{P}_{imn}(\boldsymbol{k}_0)\,\widetilde{u}^{(0)}_m(-\boldsymbol{p}_0,t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{G}^{(E0)}_{nj}(-\boldsymbol{q}_0,t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\
&\qquad -\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0}\,\mathrm{i}\left(\frac{2\pi}{L}\right)^3 \widetilde{P}_{imn}(\boldsymbol{k}_0)\,\widetilde{u}^{(0)}_m(-\boldsymbol{q}_0,t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{G}^{(E0)}_{nj}(-\boldsymbol{p}_0,t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\
&\qquad +\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0}\,\mathrm{i}\left(\frac{2\pi}{L}\right)^3 \widetilde{P}_{imn}(\boldsymbol{k}_0)\,\widetilde{u}^{(0)}_m(\boldsymbol{p}_0,t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{G}^{(E0)}_{nj}(\boldsymbol{q}_0,t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\
&\qquad +\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0}\,\mathrm{i}\left(\frac{2\pi}{L}\right)^3 \widetilde{P}_{imn}(\boldsymbol{k}_0)\,\widetilde{u}^{(0)}_m(\boldsymbol{q}_0,t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{G}^{(E0)}_{nj}(\boldsymbol{p}_0,t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\
&\qquad +\ (\boldsymbol{p}_0\to\boldsymbol{q}_0\to\boldsymbol{k}_0\to\boldsymbol{p}_0)
\end{aligned} \tag{A 6}$$

with initial condition

$$\widetilde{G}^{(E1)}_{ij}(\boldsymbol{k},t'|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0) = 0\ , \tag{A 7}$$

where higher-order terms of the DI-field have been neglected (Assumption 1).

Next, the time evolution of the position function is derived from (2.22) and (2.23) as

$$\frac{\partial}{\partial t}\,\widetilde{\psi}^{(0)}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0) = -\mathrm{i}\,k_j\left(\frac{2\pi}{L}\right)^3 \sum_{\substack{\boldsymbol{p}\quad \boldsymbol{q}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\!\!{\sum}' \widetilde{u}_j(-\boldsymbol{p},t)\,\widetilde{\psi}^{(0)}(-\boldsymbol{q},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0) \tag{A 8}$$

with initial condition

$$\widetilde{\psi}^{(0)}(\boldsymbol{k},t'|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0) = \frac{L^3}{(2\pi)^6}\,\delta^3_{\boldsymbol{k}+\boldsymbol{k}'}\ , \tag{A 9}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t}\,\widetilde{\psi}^{(1)}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\
&\quad = -\mathrm{i}\,k_j\left(\frac{2\pi}{L}\right)^3 \sum_{\substack{\boldsymbol{p}\quad \boldsymbol{q}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\!\!{\sum}' \widetilde{u}_j(-\boldsymbol{p},t)\,\widetilde{\psi}^{(1)}(-\boldsymbol{q},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\
&\qquad -\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0}\,\mathrm{i}\,k_{0j}\left(\frac{2\pi}{L}\right)^3 \widetilde{u}^{(0)}_j(-\boldsymbol{p}_0,t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{\psi}^{(0)}(-\boldsymbol{q}_0,t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)
\end{aligned}$$

$$
-\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0}\, \mathrm{i} k_{0j} \left(\frac{2\pi}{L}\right)^3 \widetilde{u}_j^{(0)}(-\boldsymbol{q}_0, t \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)\, \widetilde{\psi}^{(0)}(-\boldsymbol{p}_0, t | \boldsymbol{k}', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)
$$

$$
+\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0}\, \mathrm{i} k_{0j} \left(\frac{2\pi}{L}\right)^3 \widetilde{u}_j^{(0)}(\boldsymbol{p}_0, t \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)\, \widetilde{\psi}^{(0)}(\boldsymbol{q}_0, t | \boldsymbol{k}', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)
$$

$$
+\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0}\, \mathrm{i} k_{0j} \left(\frac{2\pi}{L}\right)^3 \widetilde{u}_j^{(0)}(\boldsymbol{q}_0, t \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)\, \widetilde{\psi}^{(0)}(\boldsymbol{p}_0, t | \boldsymbol{k}', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)
$$

$$
+ \ (\boldsymbol{p}_0 \to \boldsymbol{q}_0 \to \boldsymbol{k}_0 \to \boldsymbol{p}_0)\ , \tag{A 10}
$$

with initial condition

$$
\widetilde{\psi}^{(1)}(\boldsymbol{k}, t' | \boldsymbol{k}', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0) = 0\ . \tag{A 11}
$$

Finally, for the position response function, we obtain, from (2.37) and (2.38),

$$
\frac{\partial}{\partial t}\, \widetilde{\Psi}_i^{(0)}(\boldsymbol{k}, t, | \boldsymbol{k}', \boldsymbol{k}'', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)
$$

$$
= -\mathrm{i} k_a \left(\frac{2\pi}{L}\right)^3 \mathop{\sum_{\boldsymbol{p}} \sum_{\boldsymbol{q}}}_{(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}{}' \widetilde{u}_a(-\boldsymbol{p}, t)\, \widetilde{\Psi}_i^{(0)}(-\boldsymbol{q}, t | \boldsymbol{k}', \boldsymbol{k}'', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)
$$

$$
-\mathrm{i} k_a \left(\frac{2\pi}{L}\right)^3 \mathop{\sum_{\boldsymbol{p}} \sum_{\boldsymbol{q}}}_{(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})} \widetilde{G}_{ai}^{(E)}(-\boldsymbol{p}, t | \boldsymbol{k}'', t')\, \widetilde{\psi}(-\boldsymbol{q}, t | \boldsymbol{k}', t') \tag{A 12}
$$

with

$$
\widetilde{\Psi}_i^{(0)}(\boldsymbol{k}, t' | \boldsymbol{k}', \boldsymbol{k}'', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0) = 0 \tag{A 13}
$$

and

$$
\frac{\partial}{\partial t}\, \widetilde{\Psi}_i^{(1)}(\boldsymbol{k}, t, | \boldsymbol{k}', \boldsymbol{k}'', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)
$$

$$
= -\mathrm{i} k_a \left(\frac{2\pi}{L}\right)^3 \mathop{\sum_{\boldsymbol{p}} \sum_{\boldsymbol{q}}}_{(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}{}' \widetilde{u}_a(-\boldsymbol{p}, t)\, \widetilde{\Psi}_i^{(1)}(-\boldsymbol{q}, t | \boldsymbol{k}', \boldsymbol{k}'', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)
$$

$$
-\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0} \mathrm{i} k_{0a} \left(\frac{2\pi}{L}\right)^3 \widetilde{u}_a^{(0)}(-\boldsymbol{p}_0, t \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)\, \widetilde{\Psi}_i^{(0)}(-\boldsymbol{q}_0, t | \boldsymbol{k}', \boldsymbol{k}'', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)
$$

$$
-\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0} \mathrm{i} k_{0a} \left(\frac{2\pi}{L}\right)^3 \widetilde{u}_a^{(0)}(-\boldsymbol{q}_0, t \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)\, \widetilde{\Psi}_i^{(0)}(-\boldsymbol{p}_0, t | \boldsymbol{k}', \boldsymbol{k}'', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)
$$

$$
+\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0} \mathrm{i} k_{0a} \left(\frac{2\pi}{L}\right)^3 \widetilde{u}_a^{(0)}(\boldsymbol{p}_0, t \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)\, \widetilde{\Psi}_i^{(0)}(\boldsymbol{q}_0, t | \boldsymbol{k}', \boldsymbol{k}'', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)
$$

$$
+\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0} \mathrm{i} k_{0a} \left(\frac{2\pi}{L}\right)^3 \widetilde{u}_a^{(0)}(\boldsymbol{q}_0, t \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)\, \widetilde{\Psi}_i^{(0)}(\boldsymbol{p}_0, t | \boldsymbol{k}', \boldsymbol{k}'', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0)
$$

$$
+ \ (\boldsymbol{p}_0 \to \boldsymbol{q}_0 \to \boldsymbol{k}_0 \to \boldsymbol{p}_0) \tag{A 14}
$$

with

$$
\widetilde{\Psi}_i^{(1)}(\boldsymbol{k}, t' | \boldsymbol{k}', \boldsymbol{k}'', t' \| \boldsymbol{k}_0, \boldsymbol{p}_0, \boldsymbol{q}_0) = 0\ . \tag{A 15}
$$

In the derivation of (A 10) and (A 14), higher-order terms of the DI-field have been neglected (Assumption 1).

Appendix B

In performing Procedure 2, the DI-fields must be represented in terms of the NDI-fields. Here, we will give the expressions for $\widetilde{G}_{ij}^{(E1)}$, $\widetilde{\psi}^{(1)}$ and $\widetilde{\Psi}_i^{(1)}$.

First, by using (A 4) and (A 7), we can solve (A 6) to obtain

$$\begin{aligned}\widetilde{G}_{ij}^{(E1)}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0) = \mathrm{i}\frac{(2\pi)^9}{L^6}\widetilde{P}_{abc}(\boldsymbol{k})\int_{t'}^{t}\mathrm{d}t''\widetilde{G}_{ia}^{(E0)}(\boldsymbol{k},t|-\boldsymbol{k},t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ \times\Big[-\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0}\widetilde{u}_b^{(0)}(-\boldsymbol{p}_0,t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\widetilde{G}_{cj}^{(E0)}(-\boldsymbol{q}_0,t''|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ -\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0}\widetilde{u}_b^{(0)}(-\boldsymbol{q}_0,t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\widetilde{G}_{cj}^{(E0)}(-\boldsymbol{p}_0,t''|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ -\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0}\widetilde{u}_b^{(0)}(\boldsymbol{p}_0,t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\widetilde{G}_{cj}^{(E0)}(\boldsymbol{q}_0,t''|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ -\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0}\widetilde{u}_b^{(0)}(\boldsymbol{q}_0,t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\widetilde{G}_{cj}^{(E0)}(\boldsymbol{p}_0,t''|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ +(\boldsymbol{p}_0\to\boldsymbol{q}_0\to\boldsymbol{k}_0\to\boldsymbol{p}_0)\Big].\end{aligned} \tag{B 1}$$

Next, it follows from (A 8), (A 10) and (A 11) that

$$\begin{aligned}\widetilde{\psi}^{(1)}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0) = \mathrm{i}\,k_j\,\frac{(2\pi)^9}{L^6}\int_{t'}^{t}\mathrm{d}t''\,\widetilde{\psi}^{(0)}(\boldsymbol{k},t|-\boldsymbol{k},t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ \times\Big[-\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0}\,\widetilde{u}_j^{(0)}(-\boldsymbol{p}_0,t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{\psi}^{(0)}(-\boldsymbol{q}_0,t''|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ -\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0}\,\widetilde{u}_j^{(0)}(-\boldsymbol{q}_0,t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{\psi}^{(0)}(-\boldsymbol{p}_0,t''|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ -\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0}\,\widetilde{u}_j^{(0)}(\boldsymbol{p}_0,t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{\psi}^{(0)}(\boldsymbol{q}_0,t''|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ -\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0}\,\widetilde{u}_j^{(0)}(\boldsymbol{q}_0,t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{\psi}^{(0)}(\boldsymbol{p}_0,t''|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ +\,(\boldsymbol{p}_0\to\boldsymbol{q}_0\to\boldsymbol{k}_0\to\boldsymbol{p}_0)\Big].\end{aligned} \tag{B 2}$$

Finally, it is shown from (A 12), (A 14) and (A 15) that

$$\begin{aligned}&\widetilde{\Psi}_i^{(1)}(\boldsymbol{k},t|\boldsymbol{k}',\boldsymbol{k}'',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ &= \mathrm{i}\,k_a\frac{(2\pi)^9}{L^6}\int_{t'}^{t}\mathrm{d}t''\,\widetilde{\psi}^{(0)}(\boldsymbol{k},t|-\boldsymbol{k},t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ &\times\Big[-\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0}\,\widetilde{u}_a^{(0)}(-\boldsymbol{p}_0,t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{\Psi}_i^{(0)}(-\boldsymbol{q}_0,t''|\boldsymbol{k}',\boldsymbol{k}'',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ &-\delta^3_{\boldsymbol{k}-\boldsymbol{k}_0}\,\widetilde{u}_a^{(0)}(-\boldsymbol{q}_0,t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{\Psi}_i^{(0)}(-\boldsymbol{p}_0,t''|\boldsymbol{k}',\boldsymbol{k}'',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ &-\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0}\,\widetilde{u}_a^{(0)}(\boldsymbol{p}_0,t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{\Psi}_i^{(0)}(\boldsymbol{q}_0,t''|\boldsymbol{k}',\boldsymbol{k}'',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ &-\delta^3_{\boldsymbol{k}+\boldsymbol{k}_0}\,\widetilde{u}_a^{(0)}(\boldsymbol{q}_0,t''\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{\Psi}_i^{(0)}(\boldsymbol{p}_0,t''|\boldsymbol{k}',\boldsymbol{k}'',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\\ &+\,(\boldsymbol{p}_0\to\boldsymbol{q}_0\to\boldsymbol{k}_0\to\boldsymbol{p}_0)\Big].\end{aligned} \tag{B 3}$$

Appendix C

We derive here relations (3.5)–(3.8). First, by taking an ensemble average of (A 8) and using Assumption 3, we obtain

$$\frac{\partial}{\partial t}\,\overline{\widetilde{\psi}^{(0)}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)} = 0\;, \tag{C 1}$$

which leads to (3.6) under initial condition (A 9) of $\widetilde{\psi}^{(0)}$. Next, an ensemble average of (2.39) gives, under Assumptions 1 and 3, that

$$\begin{aligned}&\overline{\widetilde{G}^{(E0)}_{ij}(\boldsymbol{k},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}\\&\quad=\frac{(2\pi)^6}{L^3}\sum_{\boldsymbol{k}''}\overline{\widetilde{G}^{(L0)}_{ij}(t|\boldsymbol{k}'',\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}\;\overline{\widetilde{\psi}^{(0)}(\boldsymbol{k},t|-\boldsymbol{k}'',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}\;,\end{aligned} \tag{C 2}$$

where $\widetilde{G}^{(L0)}_{ij}$ denotes the NDI-field of $\widetilde{G}^{(L)}_{ij}$. By substituting (3.6) into the above equation, we obtain (3.5). For (3.7), we take an ensemble average of (A 12) to obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\,\overline{\widetilde{\Psi}^{(0)}_i(\boldsymbol{k},t,|\boldsymbol{k}',\boldsymbol{k}'',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}\\&\quad=-\mathrm{i}\,k_a\left(\frac{2\pi}{L}\right)^3\sum_{\boldsymbol{p}}\sum_{\substack{\boldsymbol{q}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\overline{\widetilde{G}^{(E0)}_{ai}(-\boldsymbol{p},t|\boldsymbol{k}'',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}\;\overline{\widetilde{\psi}^{(0)}(-\boldsymbol{q},t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}\;,\end{aligned} \tag{C 3}$$

where Assumptions 1 and 3 have been employed. Equation (3.7) follows by substituting (3.5) and (3.6) into (C 3) and integrating it under initial condition (A 13). Finally, in order to show (3.8) we note the identity

$$\begin{aligned}&\overline{\widetilde{u}^{(0)}_i(\boldsymbol{k},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{u}^{(0)}_j(-\boldsymbol{k},t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)} = \widetilde{P}_{ia}(\boldsymbol{k})\,\frac{(2\pi)^6}{L^3}\\&\quad\times\sum_{\boldsymbol{k}'}\overline{\widetilde{v}^{(0)}_a(t|\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{\psi}^{(0)}(\boldsymbol{k},t|-\boldsymbol{k}',t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{u}^{(0)}_j(-\boldsymbol{k},t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}\;,\end{aligned} \tag{C 4}$$

which follows from continuity equation (2.21) and relation (2.19) between $\widetilde{u}_i$ and $\widetilde{v}_j$. Here, $\widetilde{v}^{(0)}_a$ denotes the NDI-field of the Lagrangian velocity. By substituting (3.6) and replacing ($\widetilde{u}^{(0)}_i$, $\widetilde{v}^{(0)}_i$) by ($\widetilde{u}_i$, $\widetilde{v}_i$) (Assumption 1), we obtain

$$\overline{\widetilde{u}^{(0)}_i(\boldsymbol{k},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{u}^{(0)}_j(-\boldsymbol{k},t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)} = \widetilde{P}_{ia}(\boldsymbol{k})\,\overline{\widetilde{v}_a(t|\boldsymbol{k},t')\,\widetilde{u}_j(-\boldsymbol{k},t')}\;. \tag{C 5}$$

Then, (3.8) follows from (C 5), (2.8) and (2.29).

Appendix D

The deduction of (3.11) from (3.10) is described here. The first term on the right-hand side of (3.10) vanishes because of Assumption 2. On substitution of

$$\begin{aligned}&\widetilde{u}^{(1)}_n(-\boldsymbol{q},t\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\\&\quad=\mathrm{i}\frac{(2\pi)^9}{L^6}\,\widetilde{P}_{abc}(\boldsymbol{q})\int_{t_0}^{t}\mathrm{d}t'\,\widetilde{G}^{(E0)}_{na}(-\boldsymbol{q},t|\boldsymbol{q},t'\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\,\widetilde{u}^{(0)}_b(\boldsymbol{p},t'\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\,\widetilde{u}^{(0)}_c(\boldsymbol{k},t'\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}),\end{aligned} \tag{D 1}$$

which is derived from (3.4), in the second term of (3.10), we obtain

The second term on the right-hand side of (3.10)

$$= \frac{(2\pi)^{15}}{L^{12}} \widetilde{P}_{imn}(\boldsymbol{k}) \sum_{\substack{\boldsymbol{p} \quad \boldsymbol{q} \\ (\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\sum \widetilde{P}_{abc}(\boldsymbol{q}) \int_{t_0}^{t} \mathrm{d}t' \, \overline{\widetilde{G}^{(E0)}_{na}(-\boldsymbol{q}, t|\boldsymbol{q}, t'\|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})}$$
$$\overline{\times \widetilde{u}^{(0)}_m(-\boldsymbol{p}, t\|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})\, \widetilde{u}^{(0)}_b(\boldsymbol{p}, t'\|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})\, \widetilde{u}^{(0)}_j(-\boldsymbol{k}, t\|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})\, \widetilde{u}^{(0)}_c(\boldsymbol{k}, t'\|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})}. \quad \text{(D 2)}$$

Assumptions 2 and 3, (3.5) and (3.8) then convert (D 2) into

The second term on the right-hand side of (3.10)

$$= \frac{(2\pi)^{15}}{L^{12}} \widetilde{P}_{imn}(\boldsymbol{k}) \sum_{\substack{\boldsymbol{p} \quad \boldsymbol{q} \\ (\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\sum \widetilde{P}_{abc}(\boldsymbol{q}) \int_{t_0}^{t} \mathrm{d}t' \, \overline{\widetilde{G}^{(E0)}_{na}(-\boldsymbol{q}, t|\boldsymbol{q}, t'\|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})}$$
$$\times \overline{\widetilde{u}^{(0)}_m(-\boldsymbol{p}, t\|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})\, \widetilde{u}^{(0)}_b(\boldsymbol{p}, t'\|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})}\;\; \overline{\widetilde{u}^{(0)}_j(-\boldsymbol{k}, t\|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})\, \widetilde{u}^{(0)}_c(\boldsymbol{k}, t'\|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})}$$
$$= \frac{(2\pi)^{9}}{L^{6}} \widetilde{P}_{imn}(\boldsymbol{k}) \sum_{\substack{\boldsymbol{p} \quad \boldsymbol{q} \\ (\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\sum \widetilde{P}_{abc}(\boldsymbol{q})$$
$$\times \int_{t_0}^{t} \mathrm{d}t' \, \overline{\widetilde{G}^{(L0)}_{na}(t|-\boldsymbol{q}, \boldsymbol{q}, t'\|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})}\, \widetilde{Q}_{mb}(-\boldsymbol{p}, t, t')\, \widetilde{Q}_{jc}(-\boldsymbol{k}, t, t')\,. \quad \text{(D 3)}$$

Similarly, the third term reduces to

The third term on the right-hand side of (3.10)

$$= \frac{1}{2}\frac{(2\pi)^{9}}{L^{6}} \widetilde{P}_{imn}(\boldsymbol{k}) \sum_{\substack{\boldsymbol{p} \quad \boldsymbol{q} \\ (\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\sum \widetilde{P}_{abc}(\boldsymbol{k})$$
$$\times \int_{t_0}^{t} \mathrm{d}t' \, \overline{\widetilde{G}^{(L0)}_{ja}(t|-\boldsymbol{k}, \boldsymbol{k}, t'\|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})}\, \widetilde{Q}_{mb}(-\boldsymbol{p}, t, t')\, \widetilde{Q}_{nc}(-\boldsymbol{q}, t, t')\,. \quad \text{(D 4)}$$

Combination of (D 3) and (D 4) leads to

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] \widetilde{V}_{ij}(\boldsymbol{k}, t, t) = \frac{1}{2}\frac{(2\pi)^{9}}{L^{6}} \widetilde{P}_{imn}(\boldsymbol{k}) \sum_{\substack{\boldsymbol{p} \quad \boldsymbol{q} \\ (\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\sum \int_{t_0}^{t} \mathrm{d}t' \, \widetilde{Q}_{mb}(-\boldsymbol{p}, t, t')$$
$$\times \left\{ 2\widetilde{P}_{abc}(\boldsymbol{q})\, \overline{\widetilde{G}^{(L0)}_{na}(t|-\boldsymbol{q}, \boldsymbol{q}, t'|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})}\, \widetilde{Q}_{jc}(-\boldsymbol{k}, t, t') \right.$$
$$\left. + \widetilde{P}_{abc}(\boldsymbol{k})\, \overline{\widetilde{G}^{(L0)}_{ja}(t|-\boldsymbol{k}, \boldsymbol{k}, t'\|\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})}\, \widetilde{Q}_{nc}(-\boldsymbol{q}, t, t') \right\} + \ (i \leftrightarrow j\ ,\ \boldsymbol{k} \rightarrow -\boldsymbol{k})\,. \quad \text{(D 5)}$$

Multiplying the above equation by $\widetilde{P}_{\alpha i}(\boldsymbol{k})$, replacing suffixes appropriately and using (2.29) and (3.9), we obtain (3.11).

Appendix E

We derive here equations (3.12) and (3.13) for the two-point two-time Lagrangian velocity correlation and the response functions.

As for the two-point two-time Lagrangian velocity correlation function, the viscous term of (2.27) is expressed, under Assumption 1, as

$$\text{The viscous term of (2.27)} = -\frac{(2\pi)^9}{L^6}\nu \sum_{\boldsymbol{p}} p^2 \overline{\widetilde{u}_i^{(0)}(\boldsymbol{p},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{\psi}^{(0)}(-\boldsymbol{p},t|\boldsymbol{k},t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)\,\widetilde{u}_j^{(0)}(-\boldsymbol{k},t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}, \quad \text{(E 1)}$$

which is rewritten, using Assumption 3, (3.6) and (3.8), as

$$\text{The viscous term of (2.27)} = -\nu k^2\, \widetilde{Q}_{ij}(\boldsymbol{k},t,t') \,. \quad \text{(E 2)}$$

The nonlinear term of (2.27) is approximated, under Assumption 1, by

$$\text{The nonlinear term of (2.27)} = -\mathrm{i}\,\frac{(2\pi)^{12}}{L^9} \sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{r}}_{(\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0})} \frac{r_i r_m r_n}{r^2}$$
$$\times \Bigg[\, \overline{\widetilde{u}_m^{(0)}(\boldsymbol{p},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{u}_n^{(0)}(\boldsymbol{q},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{\psi}^{(0)}(\boldsymbol{r},t|\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{u}_j^{(0)}(-\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})}$$
$$+2\overline{\widetilde{u}_m^{(1)}(\boldsymbol{p},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{u}_n^{(0)}(\boldsymbol{q},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{\psi}^{(0)}(\boldsymbol{r},t|\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{u}_j^{(0)}(-\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})}$$
$$+\overline{\widetilde{u}_m^{(0)}(\boldsymbol{p},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{u}_n^{(0)}(\boldsymbol{q},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{\psi}^{(0)}(\boldsymbol{r},t|\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{u}_j^{(1)}(-\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})}$$
$$+\overline{\widetilde{u}_m^{(0)}(\boldsymbol{p},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{u}_n^{(0)}(\boldsymbol{q},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{\psi}^{(1)}(\boldsymbol{r},t|\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{u}_j^{(0)}(-\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})}\,\Bigg]. \quad \text{(E 3)}$$

The first term of the above equation vanishes because of (3.6) and Assumptions 2 and 3. It is easily shown that both the second and third terms are proportional to k_i. By substituting (B 2) in the fourth term, and using Assumptions 2, 3, (3.6) and (3.8), we obtain

$$\text{The fourth term of (E 3)}$$
$$= -2\left(\frac{2\pi}{L}\right)^3 \sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}}_{(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})} \frac{q_a q_m q_n q_i}{q^2} \int_{t'}^{t} \mathrm{d}t''\, \widetilde{Q}_{ma}(\boldsymbol{p},t,t'')\,\widetilde{Q}_{nj}(\boldsymbol{k},t,t') \,. \quad \text{(E 4)}$$

Therefore, (2.27) reduces to

$$\frac{\partial}{\partial t}\widetilde{V}_{ij}(\boldsymbol{k},t,t') = -\,\nu k^2\,\widetilde{Q}_{ij}(\boldsymbol{k},t,t')$$
$$-\,2\left(\frac{2\pi}{L}\right)^3 \sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}}_{(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})} \frac{q_a q_m q_n q_i}{q^2} \int_{t'}^{t} \mathrm{d}t''\, \widetilde{Q}_{ma}(\boldsymbol{p},t,t'')\,\widetilde{Q}_{nj}(\boldsymbol{k},t,t')$$
$$+\ \text{terms which are proportional to } k_i \,. \quad \text{(E 5)}$$

By multiplying the above equation by $\widetilde{P}_{li}(\boldsymbol{k})$, and noting that $\widetilde{P}_{li}(\boldsymbol{k})\,k_i = 0$ and (2.29), we arrive at (3.12).

For the Lagrangian velocity response function we have only to deal with an ensemble average of (2.35) for $\boldsymbol{k}' = -\boldsymbol{k}$, i.e.

$$\frac{\partial}{\partial t}\overline{\widetilde{G}^{(L)}_{ij}(t|\boldsymbol{k},-\boldsymbol{k},t')} = -\nu\,\frac{(2\pi)^6}{L^3} \times \sum_{\boldsymbol{k}''} k''^2 \left[\overline{\widetilde{G}^{(E)}_{ij}(\boldsymbol{k}'',t|-\boldsymbol{k},t')\,\widetilde{\psi}(-\boldsymbol{k}'',t|\boldsymbol{k},t')} + \overline{\widetilde{u}_i(\boldsymbol{k}'',t)\,\widetilde{\Psi}_j(-\boldsymbol{k}'',t|\boldsymbol{k},-\boldsymbol{k},t')}\right]$$
$$-\mathrm{i}\,\frac{(2\pi)^9}{L^6}\sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{r}\atop(\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0})}\frac{r_i r_m r_n}{r^2} \times\left[2\overline{\widetilde{u}_m(\boldsymbol{p},t)\,\widetilde{G}^{(E)}_{nj}(\boldsymbol{q},t|-\boldsymbol{k},t')\,\widetilde{\psi}(\boldsymbol{r},t|\boldsymbol{k},t')} + \overline{\widetilde{u}_m(\boldsymbol{p},t)\,\widetilde{u}_n(\boldsymbol{q},t)\,\widetilde{\Psi}_j(\boldsymbol{r},t|\boldsymbol{k},-\boldsymbol{k},t')}\right], \quad \text{(E 6)}$$

because only this combination appears in the equations for the correlation function (3.11) and (3.12). The viscous term of this equation is calculated, using Assumptions 1 and 3, (3.5) and (3.6), to be

$$\text{The viscous term of (E 6)} = -\nu\,\frac{(2\pi)^6}{L^3}\sum_{\boldsymbol{k}''}k''^2\,\overline{\widetilde{G}^{(E0)}_{ij}(\boldsymbol{k}'',t|-\boldsymbol{k},t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}\;\overline{\widetilde{\psi}^{(0)}(-\boldsymbol{k}'',t|\boldsymbol{k},t'\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}$$
$$= -\nu\,k^2\,\overline{\widetilde{G}^{(L0)}_{ij}(t|\boldsymbol{k},-\boldsymbol{k},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)}\,. \quad \text{(E 7)}$$

The first term in the second brackets of (E 6) is rewritten as

$$\text{The first term in the second brackets of (E 6)} = -2\mathrm{i}\frac{(2\pi)^9}{L^6}\sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{r}\atop(\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0})}\frac{r_i r_m r_n}{r^2}$$
$$\times\left[\overline{\widetilde{u}^{(0)}_m(\boldsymbol{p},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{G}^{(E0)}_{nj}(\boldsymbol{q},t|-\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{\psi}^{(0)}(\boldsymbol{r},t|\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})}\right.$$
$$+\overline{\widetilde{u}^{(1)}_m(\boldsymbol{p},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{G}^{(E0)}_{nj}(\boldsymbol{q},t|-\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{\psi}^{(0)}(\boldsymbol{r},t|\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})}$$
$$+\overline{\widetilde{u}^{(0)}_m(\boldsymbol{p},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{G}^{(E1)}_{nj}(\boldsymbol{q},t|-\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{\psi}^{(0)}(\boldsymbol{r},t|\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})}$$
$$\left.+\overline{\widetilde{u}^{(0)}_m(\boldsymbol{p},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{G}^{(E0)}_{nj}(\boldsymbol{q},t|-\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{\psi}^{(1)}(\boldsymbol{r},t|\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})}\right], \quad \text{(E 8)}$$

where higher-order terms of the DI-field have been neglected (Assumption 1). Thanks to Assumption 3 and $\overline{\widetilde{u}^{(0)}_i} = 0$, the first term of (E 8) vanishes. For the other terms, we employ the procedures described in §3.2. On substitution of (3.4) in the second term to eliminate $\widetilde{u}^{(1)}_m$, we can show that it vanishes because of Assumptions 2 and 3. For the third term, we use (B 1) to eliminate $\widetilde{G}^{(E1)}_{nj}$. Then, Assumption 3, (3.5), (3.6) and (3.8) reduce it to

$$\text{The third term of (E 8)} = -2\,\frac{(2\pi)^9}{L^6}\,\frac{k_i k_m k_n}{k^2} \sum_{\substack{\boldsymbol{p}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\sum_{\boldsymbol{q}} \widetilde{P}_{abc}(\boldsymbol{q})$$

$$\times \int_{t'}^{t} \mathrm{d}t''\, \overline{\widetilde{G}^{(L0)}_{na}(t|-\boldsymbol{q},\boldsymbol{q},t''\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})}\, \overline{\widetilde{G}^{(L0)}_{cj}(t''|\boldsymbol{k},-\boldsymbol{k},t'\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})}\, \widetilde{Q}_{mb}(-\boldsymbol{p},t,t''). \quad \text{(E 9)}$$

For the fourth term, (B 2) is used to eliminate $\widetilde{\psi}^{(1)}$. Then, we can rewrite it, using Assumptions 2 and 3, (3.5), (3.6) and (3.8), as

$$\text{The fourth term of (E 8)} = -2\left(\frac{2\pi}{L}\right)^3 \sum_{\substack{\boldsymbol{p}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\sum_{\boldsymbol{q}} \frac{q_i q_m q_n q_a}{q^2}$$

$$\times \int_{t'}^{t} \mathrm{d}t''\, \overline{\widetilde{G}^{(L0)}_{nj}(t|\boldsymbol{k},-\boldsymbol{k},t'\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})}\; \widetilde{Q}_{ma}(\boldsymbol{p},t,t'')\,. \quad \text{(E 10)}$$

Finally, we calculate the second term in the second brackets of (E 6). By neglecting higher-order terms of the DI-field under Assumption 1, we obtain

$$\text{The second term in the second brackets of (E 6)} = -\mathrm{i}\,\frac{(2\pi)^9}{L^6} \sum_{\substack{\boldsymbol{p}\\(\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}=\boldsymbol{0})}}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{r}} \frac{r_i r_m r_n}{r^2}$$

$$\times \Bigg[\, \overline{\widetilde{u}^{(0)}_m(\boldsymbol{p},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{u}^{(0)}_n(\boldsymbol{q},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{\Psi}^{(0)}_j(\boldsymbol{r},t|\boldsymbol{k},-\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})}$$

$$+2\,\overline{\widetilde{u}^{(1)}_m(\boldsymbol{p},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{u}^{(0)}_n(\boldsymbol{q},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{\Psi}^{(0)}_j(\boldsymbol{r},t|\boldsymbol{k},-\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})}$$

$$+\,\overline{\widetilde{u}^{(0)}_m(\boldsymbol{p},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{u}^{(0)}_n(\boldsymbol{q},t\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\widetilde{\Psi}^{(1)}_j(\boldsymbol{r},t|\boldsymbol{k},-\boldsymbol{k},t'\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})}\,\Bigg]\,. \quad \text{(E 11)}$$

The first term of this equation vanishes because $\widetilde{u}^{(0)}_m(\boldsymbol{p}\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})$ has no correlation with $\widetilde{u}^{(0)}_n(\boldsymbol{q}\|\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})$ (Assumption 2). Next, we substitute (3.4) into the second term, and (B 3) into the third term to eliminate quantities of the DI-field. Then, it is easy to show that these terms vanish under Assumptions 2 and 3. The second term of the second brackets of (E 6), therefore, does not contribute at all to the governing equation of the Lagrangian velocity response function. Combination of (E 7), (E 9) and (E 10) converts (E 6) into

$$\frac{\partial}{\partial t}\,\overline{\widetilde{G}^{(L)}_{ij}(t|\boldsymbol{k},-\boldsymbol{k},t')} + \nu\, k^2\, \overline{\widetilde{G}^{(L0)}_{ij}(t|\boldsymbol{k},-\boldsymbol{k},t\|\boldsymbol{k}_0,\boldsymbol{p}_0,\boldsymbol{q}_0)} = -2\,\frac{(2\pi)^9}{L^6}\,\frac{k_i k_m k_n}{k^2} \sum_{\substack{\boldsymbol{p}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\sum_{\boldsymbol{q}} \widetilde{P}_{abc}(\boldsymbol{q})$$

$$\times \int_{t'}^{t} \mathrm{d}t''\, \overline{\widetilde{G}^{(L0)}_{na}(t|-\boldsymbol{q},\boldsymbol{q},t''\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})}\; \overline{\widetilde{G}^{(L0)}_{cj}(t''|\boldsymbol{k},-\boldsymbol{k},t'\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})}\; \widetilde{Q}_{mb}(-\boldsymbol{p},t,t'')$$

$$-2\left(\frac{2\pi}{L}\right)^3 \sum_{\substack{\boldsymbol{p}\\(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0})}}\sum_{\boldsymbol{q}} \frac{q_i q_m q_n q_a}{q^2} \int_{t'}^{t} \mathrm{d}t''\, \overline{\widetilde{G}^{(L0)}_{nj}(t|\boldsymbol{k},-\boldsymbol{k},t'\|\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})}\; \widetilde{Q}_{ma}(\boldsymbol{p},t,t'')\,. \quad \text{(E 12)}$$

By multiplying this equation by $\widetilde{P}_{j\alpha}(\boldsymbol{k})$ and using (3.9), we obtain (3.13).

Appendix F

We assume that in the universal range of decaying turbulence $Q(k,t,t)$ is characterized by k, $\epsilon(t)$ and ν, and therefore written as

$$Q(k,t,t) = \nu^{11/4}\,\epsilon(t)^{-1/4}\,\widehat{Q}(\mathscr{B}\,\nu^{4/3}\,\epsilon(t)^{-1/4}\,k\,)\,, \tag{F 1}$$

where $\mathscr{B}$ is a non-dimensional constant. Then, it can be shown, under the assumption of a power decay law (5.10) of energy, that in (3.22) the time-derivative term is smaller in magnitude than the viscous and the nonlinear terms by $Re^{-1/2}$. Hence, in the limit of $Re \to \infty$, $Q(k,t,t)$ is independent of time in this range, which also allows a stationary form such as $G(k,t,t') = \check{G}(k,t-t')$ in the governing equation (3.16) of $G(k,t,t')$. In conclusion, the solution for decaying turbulence in this range is identical to that for stationary turbulence.

Appendix G

Equation (5.37) is derived here. Successive changes of variables in (5.22), t' to tt', t to $t^{2/3}$ and (pt,qt) to (p,q), lead to

$$\begin{aligned} t^{4/3b-2}\,\frac{\partial}{\partial t}\left(t^{4/3-4/3b}\,E^{\ddagger}(t^{2/3})\right) &= \tfrac{2}{3}\,t^3 \iint_{\triangle_t} \mathrm{d}p\mathrm{d}q\; pq\,\widehat{b}(t,p,q) \int_0^1 \mathrm{d}t'\; t'^{3-3/b} \\ &\quad \times G^{\ddagger}(t^{2/3},t^{2/3}t')\,G^{\ddagger}(p^{2/3},p^{2/3}t')\,G^{\ddagger}(q^{2/3},q^{2/3}t')\,q^{-11/3}\,E^{\ddagger}(q^{2/3}t') \\ &\quad \times\left[p^{11/3}\,E^{\ddagger}(p^{2/3}t') - t^{11/3}\,E^{\ddagger}(t^{2/3}t')\right]. \end{aligned} \tag{G 1}$$

By integrating both sides of the above equation from 0 to infinity with respect to t, we obtain

$$\mathrm{LHS} = -\left(\frac{2}{b} - 3\right)\int_0^{\infty} \mathrm{d}t\; t^{-2}\,E^{\ddagger}(t)\,, \tag{G 2}$$

$$\begin{aligned} \mathrm{RHS} = &-\lim_{T\to\infty}\,\tfrac{2}{3}\,T^{2/3}\int_1^{\infty}\mathrm{d}t\int_0^{\infty}\mathrm{d}p\int_{\max\{t-p,p\}}^{t+p}\mathrm{d}q\; t^3pq\int_0^1\mathrm{d}t'\; t'^{3-3/b} \\ &\times G^{\ddagger}((tT)^{2/3},(tT)^{2/3}t')\,G^{\ddagger}((pT)^{2/3},(pT)^{2/3}t')\,G^{\ddagger}((qT)^{2/3},(qT)^{2/3}t') \\ &\times\Bigg\{\left[\widehat{b}(t,p,q)+\widehat{b}(t,q,p)\right](pq)^{-11/3}\,E^{\ddagger}((pT)^{2/3}t')\,E^{\ddagger}((qT)^{2/3}t') \\ &-\left[\widehat{b}(t,p,q)\,q^{-11/3}\,E^{\ddagger}((qT)^{2/3}t') + \widehat{b}(t,q,p)\,p^{-11/3}\,E^{\ddagger}((pT)^{2/3}t')\right] \\ &\times t^{-11/3}\,E^{\ddagger}((tT)^{2/3}t')\Bigg\}\,. \end{aligned} \tag{G 3}$$

By taking account of (5.27) and (5.34), we can calculate (G 3) to be

$$\begin{aligned} \mathrm{RHS} = -\frac{2}{3}\int_1^{\infty}\mathrm{d}t\int_0^{1}\mathrm{d}p\int_{\max\{t-p,p\}}^{t+p}\mathrm{d}q\; t^3pq\int_0^{\infty}\mathrm{d}t'\; G^{\ddagger}_{\infty}(t^{2/3}t')\,G^{\ddagger}_{\infty}(p^{2/3}t')\,G^{\ddagger}_{\infty}(q^{2/3}t') \\ \times\left\{\left[\widehat{b}(t,p,q)+\widehat{b}(t,q,p)\right](pq)^{-11/3} - \left[\widehat{b}(t,p,q)\,q^{-11/3}+\widehat{b}(t,q,p)\,p^{-11/3}\right]t^{-11/3}\right\}. \end{aligned} \tag{G 4}$$

Equation (5.37) follows from (G 2) and (G 4).

Appendix H

For the purpose of numerical computation of (5.22)–(5.24), it is convenient to introduce H by

$$G^{\ddagger}(t,t') = H(t-t',t') \; , \qquad \text{(H 1)}$$

because the difference between two times in $G^{\ddagger}$ has the more important meaning of the response time. Substituting (H 1) into (5.22) and (5.23), we obtain

$$\begin{aligned} E^{\ddagger}(t) = &-\tfrac{9}{4}\, t^{-2+2/b} \int_0^{\infty} \mathrm{d}k \int_{2^{-2/3}}^{\infty} \mathrm{d}q \int_{|q^{3/2}-k^{3/2}|^{2/3}}^{q} \mathrm{d}p \int_{\max\left\{0,\frac{t-k}{k}\right\}}^{\infty} \mathrm{d}s\,(s+1)^{1/b-1}\, k^{8-2/b}\,(pq)^2 \\ &\times H\,(ks,k)\; H\,(ps,p)\; H\,(qs,q) \left[E^{\ddagger}(p)\, E^{\ddagger}(q)\,(pq)^{-11/2} \right. \\ &\times \left\{ \widehat{b}(k^{3/2},p^{3/2},q^{3/2}) + \widehat{b}(k^{3/2},q^{3/2},p^{3/2}) - E^{\ddagger}(k)\, k^{-11/2} \right\} \\ &\left. \times \left\{ E^{\ddagger}(p)\, p^{-11/2}\, \widehat{b}(k^{3/2},q^{3/2},p^{3/2}) + E^{\ddagger}(q)\, q^{-11/2}\, \widehat{b}(k^{3/2},p^{3/2},q^{3/2}) \right\} \right] \end{aligned} \qquad \text{(H 2)}$$

and

$$H(t,t') = \exp\left[-\int_0^{\infty} \mathrm{d}t'' \int_0^{(tt'')/(t')} \mathrm{d}s \int_{(s+t'')/(t+t')}^{(t'')/(t')} \mathrm{d}p J(p)\, p^{-2} \left[\frac{s}{t''} + 1 \right]^{1/b-1} E^{\ddagger}(t'') H(s,t'') \right], \qquad \text{(H 3)}$$

respectively. Boundary condition (5.31) is written as

$$H(0,t') = 1, \qquad \text{(H 4)}$$

while asymptotic condition (5.34) at large time is represented by

$$H(t,t') \to G^{\ddagger}_{\infty}(t) \qquad \text{as } t' \to \infty \; . \qquad \text{(H 5)}$$

Relation (5.24) between b and ζ is rewritten as

$$b = \begin{cases} \dfrac{4}{3(\zeta+3)} & (2 \leqslant \zeta < 4), \\ \left[\dfrac{21}{4} - \dfrac{7}{10\, E_0^{\ddagger}} \displaystyle\int_0^{\infty} \mathrm{d}t \int_0^{\infty} \mathrm{d}k (k+t)^{3/b-23/2} t^{3-3/b} \Big(H(k,t) E^{\ddagger}(t) \Big)^2 \right]^{-1} & (\zeta = 4). \end{cases} \qquad \text{(H 6)}$$

We solve numerically the set of integro-differential equations (H 2), (H 3) and (H 6) under boundary conditions (5.27) and (H 4), and asymptotic conditions (5.25) and (H 5). Equations (H 3) and (H 6) are solved by an iteration method, while (H 2) by the Newton–Raphson method.

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